Newton's aerodynamic problem in media of chaotically moving particles

Alexander Yu. Plakhov
plakhov@mat.ua.pt

Delfim F. M. Torres
delfim@mat.ua.pt

Department of Mathematics
University of Aveiro
3810-193 Aveiro, Portugal

Abstract

We study the problem of minimal resistance for a body moving with constant velocity in a rarefied medium of chaotically moving point particles, in Euclidean space $\mathbb{R}^d$. The particles distribution over velocities is radially symmetric. Under some additional assumptions on the distribution function, the complete classification of bodies of least resistance is made. In the case of three and more dimensions there are two kinds of solutions: a body similar to the solution of classical Newton's problem and a union of two such bodies "glued together" by rear parts of their surfaces. In the two-dimensional case there are solutions of five different types: (a) a trapezium; (b) an isosceles triangle; (c) the union of a triangle and a trapezium with common base; (d) the union of two isosceles triangles with common base; (e) the union of two triangles and a trapezium. The cases (a)–(d) are realized for any distribution of particles over velocities, and the case (e) is only realized for some distributions. Two limit cases are considered, where the average velocity of particles is big and where it is small as compared to the body's velocity. Finally, using the obtained analytical results, we study numerically a particular case: the problem of body's motion in a rarefied homogeneous monatomic ideal gas of positive temperature in $\mathbb{R}^2$ and in $\mathbb{R}^3$.

1 Introduction

In 1686, in his *Principia* [9], I. Newton considered the problem of body's motion in a homogeneous medium of point particles. He assumed that collisions of the particles with the body are absolutely elastic, the medium is very rare, so that the particles do not mutually interact, and that initially the particles are immovable, i.e., thermal motion of particles is not taken into account. These assumptions are not satisfied in the ordinary conditions "on earth", but can be approximately valid when considering motion of high-speed and high-altitude flying vehicles such as missiles and artificial satellites.

Newton considered the problem of finding the shape of body minimizing resistance of the medium to the body's motion. He solved this problem in the class of convex axially symmetric bodies with the axis parallel to the body's velocity, of fixed length along this axis and with fixed projection on a plane orthogonal to the axis. Due to convexity of the body, each particle hits the body at most once, and this fact allows one to write down an explicit analytical formula for resistance. The body of least resistance found by Newton can be described as follows: the rear part of its surface is a flat disk, which is at the same time the maximal cross section of the body by a plane orthogonal to the symmetry axis. The front part of the surface is composed of a smaller disk in the middle and of a strictly convex lateral surface.

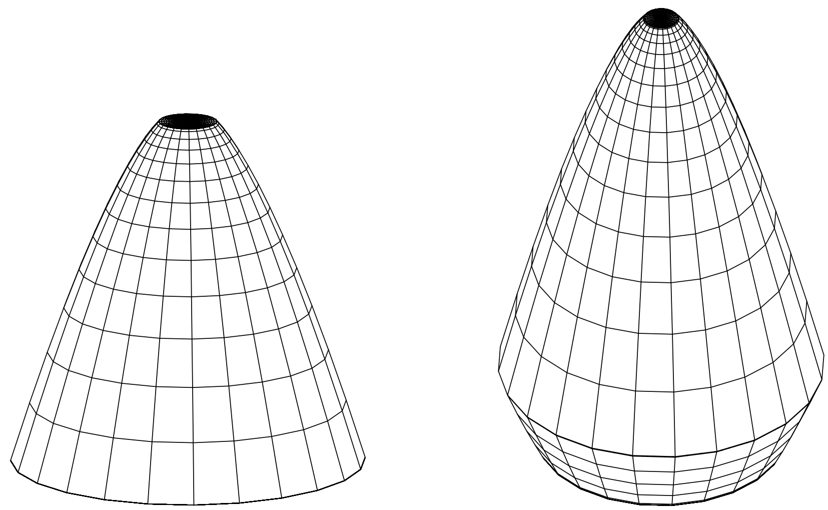
Let us also mention the two-dimensional analogue of Newton's problem. Consider a class of convex figures in $\mathbb{R}^2$ that are symmetric with respect to some straight line and have fixed length along this line and fixed width; it is required to find the figure from this class such that resistance to the motion of the figure along this line is minimal. If the length does not exceed the half-width, the solution is a trapezium with the angle 45^0 at the base; elsewhere, the solution is an isosceles triangle.

Since the early 1990th the interest to Newton's problem revived. In particular, there were obtained interesting results related to minimization problems in wider classes of bodies obtained by withdrawing or relaxing the conditions initially imposed by Newton: axial symmetry [2], [1], [7], [8] and convexity [3], [4], [5], [10], [11].

On the other hand, the assumptions of absolutely elastic collisions and of absence of thermal motion in the medium are, at the best, true only approximately. (Note that "absence of thermal motion" means that the mean velocity of thermal motion of particles is negligible as compared to the body's velocity.) In [6], the problem was studied under the more realistic hypothesis of presence of friction at the moment of collision (so that collisions are not absolutely elastic). In the present paper, we address the minimization problem in a medium with thermal noise of particles.

A convex and axisymmetric body moves in $\mathbb{R}^d,\ d \geq 2$ along its symmetry axis, in a medium of chaotically moving particles; the medium is homogeneous, and distribution of the particles over velocities is the same at every point. The magnitude V of velocity is constant. The length h of the body along the axis is fixed, and the maximal cross section of the body by a hyperplane orthogonal to the axis is a unit $(d-1)$-dimensional ball. We consider the problem of finding the shape of body minimizing resistance of the medium. The main results of this paper are as follows.

If $d \geq 3$, there are two different kinds of solutions. We shall describe them in the case $d = 3$; if $d > 3$, the description is quite similar. The solution of first kind is similar to the solution of classical Newton's problem, that is, its surface can be described in the same way as the surface of Newton's solution. The solution of second kind is a union of two bodies similar to Newton's solution "glued together" by rear parts of their surfaces. The length (along the direction of motion) of the front body is always more than the length of the rear body



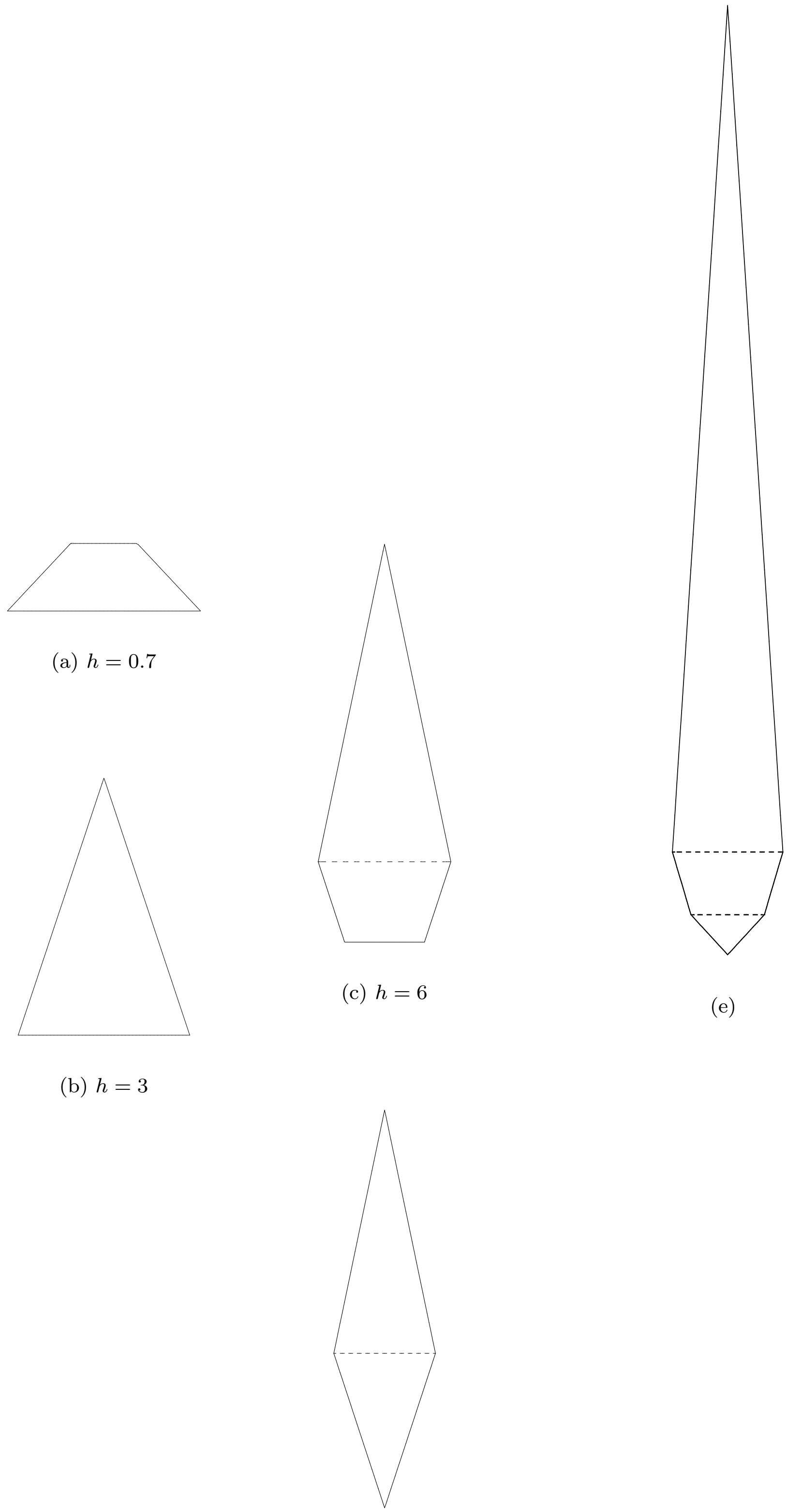
(a) $V = 1$, $h = 1.97$ (b) $V = 1$, $h = 3.11$

Figure 1: Two solutions of the three-dimensional problem related to motion in a rarefied monatomic homogeneous ideal gas. The mean square velocity of gas molecules equals 1.

turned over. The solution of first kind is realized for $h \le h_*$, and of second kind, for $h > h_*$, where $h_* = h_*(V) > 0$ is a critical value depending on V. The function $h_*(V)$ goes to infinity as $V \to +\infty$ and to zero as $V \to 0^+$. The examples of solutions of first and second kind are shown on Fig. 1(a) and on Fig 1(b), respectively. Here and in what follows the body is supposed to move vertically upwards.

If $d = 2$, the classification of solutions is somewhat more complicated. There are five different kinds of solutions: (a) a trapezium, (b) an isosceles triangle, (c) the union of a triangle and a trapezium, (d) the union of two isosceles triangles, (e) the union of two triangles and a trapezium; see Fig. 2(a) – Fig 2(e). The solutions of first kind are realized for $0 < h < u_+^0$, of second, for $u_+^0 \le h \le u_*$, of third, for $u_* < h < u_* + u_-^0$, of fourth and fifth, for $h \ge u_* + u_-^0$. These values $u_+^0 = u_+^0(V)$, $u_* = u_*(V)$, $u_-^0 = u_-^0(V)$ will be defined in section 4.2; one has $0 < u_+^0(V) < u_*(V) < u_*(V) + u_-^0(V)$. The solutions (a) – (d) are realized for any distribution of particles over velocities and for any positive V; the solution (e) is realized only for some special distributions and some values of V. The numerical computation of a solution of kind (e) is a hard task, which is unsolved as yet.

In the limit cases, where the velocity of body is big and where it is small as compared to the mean velocity of particles, the shape of body of least resistance depends only on the length h, and does not depend on the distribution of par-



(a) $h = 0.7$

(b) $h = 3$

(c) $h = 6$

(d) $h = 7.83$

(e)

Figure 2: The two-dimensional problem. The solutions corresponding to the cases (a) – (d) are calculated numerically, for the motion with velocity $V = 1$ in a gas; the gas parameters are the same as on Fig. 1.

ticles over velocities. In the first limit case the optimal body coincides with the solution of classical Newton's problem. In the second limit case, for $d = 3$, the optimal body is a second kind solution symmetric with respect to a plane perpendicular to the symmetry axis, the inclination angle of the lateral surface at its upper and lower points with respect to this plane being 51.8^0; and for $d = 2$, the optimal body is one of the four figures: (a) a trapezium if $0 < h < 1.272$; (b) an isosceles triangle if $h = 1.272$; (c) the union of an isosceles triangle and a trapezium if $1.272 < h < 2.544$; (d) a rhombus if $h \geq 2.544$. In the cases (a) – (c) the inclination angle of lateral sides of these figures with respect to the base equals 51.8^0, and in the case (d), exceeds this value.

In a monatomic ideal gas the velocities of molecules are distributed according to Gaussian law. Suppose that the mean square velocity of molecules equals 1, then the kind of solution is determined by two parameters: velocity of the body V and its length h. We define numerically the regions on the parameter plane corresponding to different kinds of solutions; for some special values of parameters we determine the shape of optimal body and calculate the corresponding resistance. This work is made in the two- and in the three-dimensional cases.

Imagine an observer travelling with the body through the medium; he would detect a flux of particles falling on the immovable body. In fact, this picture is more convenient for us, and will be taken in the sequel.

The paper is organized as follows. In section 2 the formulas for pressure of the flux on the body's surface and for resistance force are derived; two auxiliary lemmas of pressure distribution over the surface are formulated; and the problem of minimal resistance is reduced to the form more adapted for studying. In section 3, some auxiliary minimization problems are solved. Using these results, in the next section we solve the minimal resistance problem in general form. The solutions are different for the cases $d = 2$ and $d \geq 3$. In section 5, the obtained results are applied to the flux corresponding to a rarefied monatomic homogeneous ideal gas of positive temperature. In appendix A, the auxiliary lemmas are proved, and in appendix B, asymptotic formulas for pressure functions as $V \to 0^+$ are obtained.

2 Calculation of pressure and resistance

2.1 Consider an immovable convex body $\mathcal{B}$ in Euclidean space $\mathbb{R}^d$, $d \geq 2$ and a flux of infinitesimal particles falling upon it. Velocities and masses of the particles in general are different. Let the function $\rho(v)$ denote the distribution density over velocities of total mass of particles in a unit volume, so that for any two infinitesimal regions $\mathcal{V}$, $\mathcal{X} \subset \mathbb{R}^d$ having d-dimensional volumes $|\mathcal{V}| = dv$, $|\mathcal{X}| = dx$, the total mass of particles that are contained in $\mathcal{X}$ and have velocities $v \in \mathcal{V}$ equals $\rho(v)\, dv\, dx$. Therefore, the particles' distribution over velocities and density of the flux $\nu = \int_{\mathbb{R}^d} \rho(v) dv$ are the same at each point and at each instant. It is supposed that $\nu < \infty$.

An individual particle hitting the body at x transmits the impulse $m \cdot 2(v|n_x)n_x$ to the body, where m and v denote the particle's mass and its velocity

before the collision, n_x means the outer unit normal to $\partial\mathcal{B}$ at x, and $(\cdot\,|\,\cdot)$ means scalar product. Note that $(v|n_x) < 0$.

Let δ be an infinitesimal part of $\partial\mathcal{B}$ containing x, and let $\mathcal{V}$ be an infinitesimal region containing v, of volume $|\mathcal{V}| = dv$. The total mass of particles, colliding with δ in a time interval dt and having velocity $v \in \mathcal{V}$, equals $dM = \rho(v)\, dv \cdot (v|n_x)_-\, |\delta| dt$, where $|\delta|$ means $(d-1)$-dimensional area of δ, and $z_- := \max\{-z,\, 0\}$. The total impulse transmitted by these particles equals

$$dM \cdot 2(v|n_x)n_x = -2\rho(v)\, dv \cdot (v|n)_-{}^2\, |\delta| dt \cdot n_x.$$

Integrating this value with respect to v, one obtains the total impulse transmitted to δ per time dt,

$$-2 \int_{\mathbb{R}^d} (v|n_x)_-{}^2 \rho(v)\, dv \cdot |\delta| dt \cdot n_x.$$

Dividing this value by $|\delta| dt$, one obtains that pressure of the flux at x equals $\pi(n_x)$, where

$$\pi(n) = -2 \int_{\mathbb{R}^d} (v|n)_-{}^2 \rho(v)\, dv \cdot n. \tag{2.1}$$

Integrating the pressure over $\partial\mathcal{B}$, one gets the total force $R(\mathcal{B})$ the flow is exerting on the body,

$$R(\mathcal{B}) = \int_{\partial\mathcal{B}} \pi(n_x)\, d\mathcal{H}^{d-1}(x), \tag{2.2}$$

where $\mathcal{H}^{d-1}$ means $(d-1)$-dimensional Hausdorff measure.

2.2 Denote $\mathbb{R}_+ := [0,\, +\infty)$, and denote by $\mathcal{A}_d$ the set of functions $\sigma \in C^1(\mathbb{R}_+)$ such that the function $\sigma'(r)/r,\ r > 0$ is negative, bounded below, and monotone increasing, and

$$\int_0^\infty r^2 \sigma(r)\, dr^d < \infty. \tag{2.3}$$

Remark 1. Note that if $\sigma_1,\ \sigma_2 \in \mathcal{A}_d$ then $\sigma_1 + \sigma_2 \in \mathcal{A}_d$. Also, if $\alpha,\ \beta > 0$, $\sigma \in \mathcal{A}_d$ and $\tilde{\sigma}(r) = \alpha\sigma(\beta r)$, then $\tilde{\sigma} \in \mathcal{A}_d$.

From now on, we suppose that the density function ρ satisfies the condition

> **A** $\rho(v) = \sigma(|v + Ve_d|)$, where $\sigma \in \mathcal{A}_d,\ V > 0$, and e_d is the dth coordinate vector.

Note that the relation (2.3) implies that pressure $\pi(n)$ (2.1) is always finite.

Example 1. Consider a rarefied homogeneous monatomic ideal gas in $\mathbb{R}^3$ of absolute temperature $T > 0$. The distribution density of molecules' mass over velocities equals $\sigma_h(|v|)$, where

$$\sigma_h(r) = \nu \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mr^2}{2kT}}$$

(the Maxwell distribution); here k is Boltzmann's constant and ν is the gas density. Consider a body moving through the gas with constant velocity of magnitude V in the direction of third coordinate vector e_3. In a frame of reference connected with the body the distribution density over velocities equals $\rho_h(v) = \sigma_h(|v + Ve_3|)$. It is easy to check that $\sigma_h \in \mathcal{A}_3$, so the condition A is fulfilled.

Example 2. Let, now, a rarefied ideal gas of temperature T be a mixture of n homogeneous components, the ith component having density ν_i and being composed of monatomic molecules of mass m_i. Then the distribution density of molecules' mass over velocities equals $\sigma_{nh}(|v|)$, where

$$\sigma_{nh}(r) = \sum_{i=1}^{n} \nu_i \left(\frac{m_i}{2\pi kT}\right)^{3/2} e^{-\frac{m_i r^2}{2kT}} .$$

Taking account of remark 1, one concludes that $\sigma_{nh} \in \mathcal{A}_3$. As in the previous example, a body moves in the gas along the third coordinate axis with velocity V. In a frame of reference connected with the body the distribution density over velocities equals $\rho_{nh}(v) = \sigma_{nh}(|v + Ve_3|)$, therefore, the condition A is also satisfied.

In the examples 1 and 2, the force of resistance of the gas to the body's motion is calculated according to (2.1) and (2.2).

In what follows, we shall also suppose the following condition to be fulfilled:

> **B** The body $\mathcal{B}$ is convex, compact, and symmetric with respect to the dth coordinate axis. Moreover, the maximal cross section of the body by a hyperplane orthogonal to the symmetry axis is a unit $(d-1)$-dimensional ball.

By translation along the dth coordinate axis, the body can be reduced to the form

$$\mathcal{B} = \{(x', x_d) : |x'| \le 1,\ f_-(|x'|) \le x_d \le -f_+(|x'|)\} ,$$

where $x' = (x_1, \ldots, x_{d-1})$, f_+ and f_- are convex non-positive non-decreasing continuous functions defined on [0, 1]. The length h of body along the symmetry axis equals

$$h = -f_+(0) - f_-(0).$$

Now, let us specify the formulas for pressure $\pi(n)$ (2.1) and force $R(\mathcal{B})$ (2.2), using the conditions A and B.

At a regular point $x_+ = (x', -f_+(|x'|))$ of the upper part of the boundary $\partial\mathcal{B}$, the outer unit normal vector is

$$n_{x_+} = \frac{1}{\sqrt{f_+'(|x'|)^2 + 1}} \left(f_+'(|x'|)\frac{x'}{|x'|},\ 1 \right), \tag{2.4}$$

and using (2.1) and taking into account axial symmetry of ρ with respect to the dth coordinate axis, one finds that pressure of the flux at this point equals

$$\pi(n_{x_+}) = -p_+\left(f'_+(|x'|)\right)\cdot n_{x_+}, \tag{2.5}$$

where

$$p_+(u) := \left|\pi\left(\frac{1}{\sqrt{u^2+1}}\,(u,\,0,\ldots,0,\,1)\right)\right|. \tag{2.6}$$

Similarly, pressure of the flux at a regular point $x_- = (x', f_-(|x'|))$ of the lower part of $\partial\mathcal{B}$ equals

$$\pi(n_{x_-}) = p_-\left(f'_-(|x'|)\right)\cdot n_{x_-}, \tag{2.7}$$

where

$$n_{x_-} = \frac{1}{\sqrt{f'_-(|x'|)^{\,2}+1}}\ \left(f'_-(|x'|)\frac{x'}{|x'|},\ -1\right) \tag{2.8}$$

and

$$p_-(u) := -\left|\pi\left(\frac{1}{\sqrt{u^2+1}}\,(u,\,0,\ldots,0,\,-1)\right)\right|. \tag{2.9}$$

From (2.6), (2.9), and (2.1) one obtains

$$p_\varepsilon(u) = \varepsilon\int_{\mathbb{R}^d}\frac{(v_1u+\varepsilon v_d)_-^{\,2}}{1+u^2}\,\rho(v)\,dv, \quad \text{where} \quad \varepsilon\in\{-,\,+\}. \tag{2.10}$$

Let us calculate $R(\mathcal{B})$. The integral in the right hand side of (2.2) is the sum of two integrals corresponding to the upper and lower parts of $\partial\mathcal{B}$. Changing the variable in each of these integrals and using the formulas (2.4), (2.5), (2.7), and (2.8), one obtains

$$R(\mathcal{B}) = \int_{|x'|\le 1} p_+(f'_+(|x'|))\cdot\left(-f'_+(|x'|)\frac{x'}{|x'|},\ -1\right)dx' +$$
$$+\int_{|x'|\le 1} p_-(f'_-(|x'|))\cdot\left(f'_-(|x'|)\frac{x'}{|x'|},\ -1\right)dx'.$$

Next, using that the functions $p_\varepsilon(f'_\varepsilon(|x'|))$ are invariant, and the functions $\varepsilon f'_\varepsilon(|x'|)\frac{x'}{|x'|}$ are anti-invariant with respect to central symmetry $x' \to -x'$, one gets that

$$\int_{|x'|\le 1} p_\varepsilon(f'_\varepsilon(|x'|))\cdot\varepsilon f'_\varepsilon(|x'|)\,\frac{x'}{|x'|}\,dx' = 0, \qquad \varepsilon\in\{-,+\},$$

hence

$$R(\mathcal{B}) = -\left(\int_{|x'|\le 1} p_+(f'_+(|x'|))\,dx' + \int_{|x'|\le 1} p_-(f'_-(|x'|))\,dx'\right)\ e_d.$$

Therefore

$$R(\mathcal{B}) = -a_{d-1}\ (\mathcal{R}_+(f_+)+\mathcal{R}_-(f_-))\cdot e_d,$$

where a_{d-1} is the volume of a unit ball in $\mathbb{R}^{d-1}$, and

$$\mathcal{R}_\varepsilon(f) = \int_0^1 p_\varepsilon(f'(t))\, dt^{d-1}, \qquad \varepsilon \in \{-,+\}. \tag{2.11}$$

Denote by $\mathcal{M}(h)$ the class of convex non-positive non-decreasing continuous functions f defined on $[0, 1]$ such that $f(0) = -h$. Note that any function $f \in \mathcal{M}(h)$ is differentiable everywhere except possibly on a finite or countable set, and f' is monotone, hence the integral (2.11) is well defined for any $f \in \mathcal{M}(h)$.

2.3 Thus, the problem of minimal resistance takes the following form:

minimize $\mathcal{R}_+(f_+) + \mathcal{R}_-(f_-)$
provided that f_+ and f_- are convex non-positive
non-decreasing functions satisfying the relation
$-f_+(0) - f_-(0) = h$.

It will be solved in two steps. First, given $h_- \geq 0,\ h_+ \geq 0$, find

$$\inf_{f \in \mathcal{M}(h_-)} \mathcal{R}_-(f) \quad \text{ and } \quad \inf_{f \in \mathcal{M}(h_+)} \mathcal{R}_+(f). \tag{2.12}$$

Second, given solutions $f^-_{h_-},\ f^+_{h_+}$ of the problems (2.12), find

$$\mathrm{R}(h) := \inf_{h_+ + h_- = h} \left(\mathcal{R}_+(f^+_{h_+}) + \mathcal{R}_-(f^-_{h_-}) \right).$$

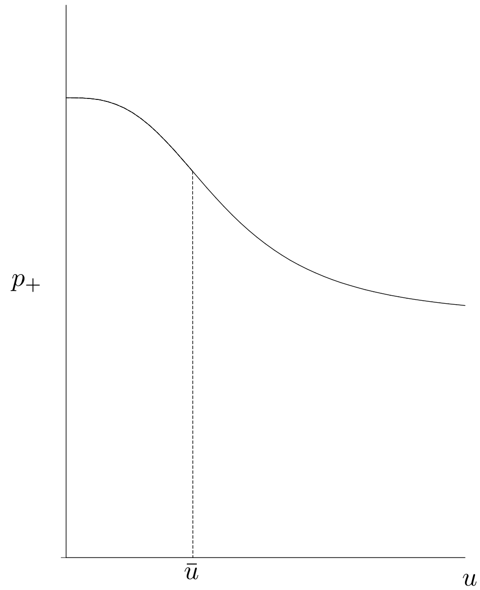
2.4 Let us formulate two auxiliary lemmas. Their proofs are rather bulky, and are given in Appendix A.

Lemma 1 states some properties of the functions $p_+,\ p_-$, which will be needed in the subsequent sections.

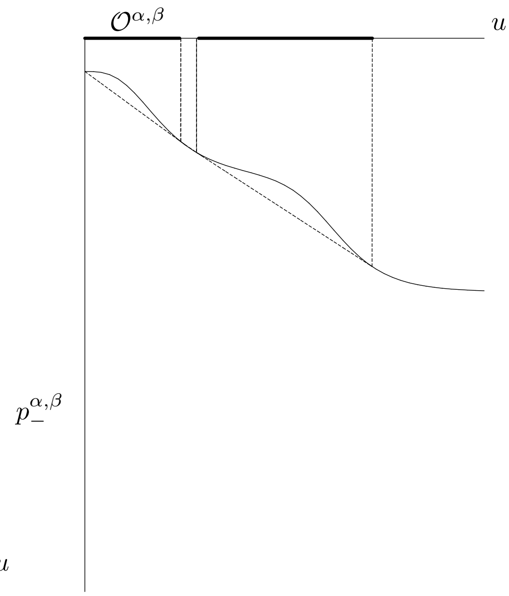
Lemma 1. *Let ρ satisfy the condition A. Then*
(a) there exist the limits $\lim_{u \to +\infty} p_\varepsilon(u) =: p_\varepsilon(+\infty)$, besides $p_+(+\infty) + p_-(+\infty) = 0$;
(b) $p_\varepsilon \in C^1(\mathbb{R}_+)$, and $p'_\varepsilon(0) = \lim_{u \to +\infty} p'_\varepsilon(u) = 0,\ \varepsilon \in \{-,+\}$;
(c) for $u > 0,\ p'_+(u) < p'_-(u)$;
(d) for $u > 0,\ p'_+(u) < 0$, and for any $u \geq 0,\ p_-(u) > p_-(+\infty)$.

Lemma 2 specifies the form of functions $p_+,\ p_-$ for $d = 2$: the function p_+ has, in a sense, a simple behavior, and the behavior of p_- may be complicated. This specification will be used in section 4.1 when constructing the body of least resistance in two dimensions.

Designate $\sigma^{\alpha,\beta}(r) = \sigma(r) + \alpha\,\sigma(\beta r)$, where $\alpha \geq 0,\ \beta > 0$. By virtue of remark 1, if $\sigma \in \mathcal{A}_2$ then $\sigma^{\alpha,\beta} \in \mathcal{A}_2$, hence the function $\rho^{\alpha,\beta}(v) = \sigma^{\alpha,\beta}(|v + Ve_2|),\ V > 0$, defined on $\mathbb{R}^2$, satisfies the condition A. Denote by $p^{\alpha,\beta}_\varepsilon,\ \varepsilon \in \{-,+\}$ the function corresponding to the density $\rho^{\alpha,\beta}$, according to the formula (2.10), and denote by $\bar{p}^{\alpha,\beta}_\varepsilon$ the maximal convex function defined on $\mathbb{R}_+$ that does not exceed $p^{\alpha,\beta}_\varepsilon$.



(a)



(b)

Figure 3:

Lemma 2. *Let $d = 2$.*

(a) If ρ satisfies the condition A then for some $\bar{u} > 0$, p'_+ is monotone decreasing on $[0, \bar{u}]$ and monotone increasing on $[\bar{u}, +\infty)$ (see Fig. 3(a)).

(b) Suppose that $\sigma \in \mathcal{A}_2$, $V > 0$, and for any $n > 0$ the function $r^n\sigma(r)$ monotonically decreases, for r large enough. Then there exist $\alpha \geq 0$, $\beta > 0$ such that the set $\mathcal{O}^{\alpha,\beta} = \{u : p_-^{\alpha,\beta}(u) > \bar{p}_-^{\alpha,\beta}(u)\}$ has at least two connected components (see Fig. 3(b); $\mathcal{O}^{\alpha,\beta}$ is shown bold-faced on the x-axis).

3 Auxiliary minimization problems

3.1 The following lemma reduces the minimization problems (2.12) to a simpler problem of minimization for a function depending on a parameter.

Lemma 3. *Let $p \in C(\mathbb{R}_+)$, $d \geq 2$, $\lambda > 0$, and let a function $f_h \in \mathcal{M}(h)$ satisfy the condition*

> **($\mathbf{C}_\lambda$)** $f_h(1) = 0$, *and for almost every t, $u = f'_h(t)$ is a solution of the problem*
> $$t^{d-2}\, p(u) + \lambda\, u \to \min\,. \tag{3.1}$$

Then f_h is a solution of the minimization problem

$$\inf_{f\in\mathcal{M}(h)} \mathcal{R}(f), \qquad \mathcal{R}(f) = \int_0^1 p(f'(t))\, dt^{d-1}. \tag{3.2}$$

Moreover, any other solution of (3.2) satisfies the condition (C_λ) with the same λ.

Proof. In fact, the problem (3.2) can be considered to be a degenerated case of the classical problem of optimal control [12], and the statement of lemma is a consequence of the Pontryagin maximum principle (see [14]). The proof we give here, however, is quite elementary and does not appeal to the maximum principle (cf. [13]).

For any $f \in \mathcal{M}(h)$ one has

$$t^{d-2}\, p(f'(t)) + \lambda\, f'(t) \geq t^{d-2}\, p(f_h'(t)) + \lambda\, f_h'(t) \tag{3.3}$$

at almost every t. Integrating both sides of (3.3) over $t \in [0,\, 1]$, one gets

$$\frac{1}{d-1}\int_0^1 p(f'(t))\, dt^{d-1} + \lambda\,(f(1) - f(0)) \geq$$

$$\geq \frac{1}{d-1}\int_0^1 p(f_h'(t))\, dt^{d-1} + \lambda\,(f_h(1) - f_h(0)), \tag{3.4}$$

and using that $f(1) \leq 0 = f_h(1)$ and $f(0) = f_h(0) = -h$, one obtains that $\mathcal{R}(f) \geq \mathcal{R}(f_h)$.

Next, suppose that $f \in \mathcal{M}(h)$ and $\mathcal{R}(f) = \mathcal{R}(f_h)$, then, using the relation (3.4) and the equality $f(0) = f_h(0)$, one gets that $f(1) \geq f_h(1) = 0$, hence $f(1) = 0$. Therefore the inequality in (3.4) becomes equality, which, in view of (3.3), implies that

$$t^{d-2}\, p(f'(t)) + \lambda\, f'(t) = t^{d-2}\, p(f_h'(t)) + \lambda\, f_h'(t)$$

for almost every t, hence $u = f'(t)$ is also a solution of (3.1), on a set of full measure. Thus, f satisfies the condition (C_λ). ◻

3.2 Assume, additionally, that $p \in C^1(\mathbb{R}_+)$ and that p is bounded below. Denote by $\bar{p}$ the maximal convex function defined on $\mathbb{R}_+$ that does not exceed p. The function $\bar{p}$ is also continuously differentiable, and for any $h \geq 0,\ u \geq 0$ one has

$$p(u) \geq \bar{p}(u) \geq \bar{p}(h) + \bar{p}'(h)\cdot(u-h). \tag{3.5}$$

Define the set $\mathcal{O}_p := \{u : \, p(u) > \bar{p}(u)\}$. Obviously, $\mathcal{O}_p$ is open, and hence is a union of a finite or countable (maybe empty) set of disjoint open intervals.

The following lemma specifies the solution f_h of the minimization problem (3.2) in the case $d = 2$.

Lemma 4. *Suppose that the function p is bounded below, $p \in C^1(\mathbb{R}_+)$, $h \geq 0$, $\bar{p}'(h) < 0$, and denote by $(h^{(-)}, h^{(+)})$ the maximal interval contained in $\mathcal{O}_p$ such that $h^{(-)} \leq h \leq h^{(+)}$ (it may happen that $h^{(-)} = h = h^{(+)}$, i.e., the interval is empty).*
Then $h^{(+)} < +\infty$, and the following holds true:

(a) The function f_h defined by

$$f_h(t) = -h + ht,$$

if $h^{(-)} = h = h^{(+)}$, and by

$$f_h(t) = \begin{cases} -h + h^{(-)}t & \text{if } t \leq t_0\,, \\ -h + h^{(-)}t_0 + h^{(+)}(t - t_0) & \text{if } t \geq t_0\,, \end{cases} \qquad \text{where } t_0 = \frac{h^{(+)} - h}{h^{(+)} - h^{(-)}},$$

if $h^{(-)} < h^{(+)}$, is a solution of the minimization problem

$$\inf_{f \in \mathcal{M}(h)} \mathcal{R}(f), \qquad \mathcal{R}(f) = \int_0^1 p(f'(t))\,dt\,, \tag{3.6}$$

besides

(b)
$$\inf_{f \in \mathcal{M}(h)} \mathcal{R}(f) = \mathcal{R}(f_h) = \bar{p}(h). \tag{3.7}$$

(c) If f is a solution of (3.6) then at almost every t, the value $u = f'(t)$ satisfies the relations $p(u) = \bar{p}(h) + \bar{p}'(h) \cdot (u - h)$ and $u \notin \mathcal{O}_p$.

Proof. p is bounded below and $\bar{p}'(h) < 0$, therefore there exists a value $u > h$ such that $\bar{p}(u) = p(u)$, hence $h^{(+)} \leq u < +\infty$.

One has $\bar{p}(h^{(-)}) = p(h^{(-)})$, $\bar{p}(h^{(+)}) = p(h^{(+)})$, and

$$p(h^{(\pm)}) = \bar{p}(h) + \bar{p}'(h) \cdot (h^{(\pm)} - h). \tag{3.8}$$

From (3.5) and (3.8) it follows that for any u

$$p(u) - p(h^{(\pm)}) \geq \bar{p}'(h) \cdot (u - h^{(\pm)}),$$

and designating $\lambda = -\bar{p}'(h)$, one obtains

$$p(u) + \lambda u \geq p(h^{(\pm)}) + \lambda h^{(\pm)}.$$

This means that both $h^{(-)}$ and $h^{(+)}$ minimize the function $p(u) + \lambda u$.

Further, one easily sees that $f_h \in \mathcal{M}(h)$, $f_h(1) = 0$, and the function f'_h takes the values $h^{(-)}$ and $h^{(+)}$ (which may coincide). Applying lemma 3, one obtains that f_h is a solution of the problem (3.6).

If $h^{(-)} \neq h^{(+)}$ then

$$\begin{aligned} \mathcal{R}(f_h) &= \int_0^{t_0} p(h^{(-)})\,dt + \int_{t_0}^1 p(h^{(+)})\,dt = \\ &= \frac{h^{(+)} - h}{h^{(+)} - h^{(-)}}\; p(h^{(-)}) + \frac{h - h^{(-)}}{h^{(+)} - h^{(-)}}\; p(h^{(+)}). \end{aligned}$$

On the other hand, excluding $\bar{p}'(h)$ from the relation (3.8), one obtains

$$\frac{h^{(+)} - h}{h^{(+)} - h^{(-)}}\, p(h^{(-)}) + \frac{h - h^{(-)}}{h^{(+)} - h^{(-)}}\, p(h^{(+)}) = \bar{p}(h),$$

hence $\mathcal{R}(f_h) = \bar{p}(h)$. If $h^{(-)} = h^{(+)} = h$ then $\mathcal{R}(f_h) = p(h) = \bar{p}(h)$, so the formula (3.7) is true.

Let, now, f be a solution of (3.6). By lemma 3, for almost every t the value $\hat{u} = f'(t)$ minimizes the function $p(u) + \lambda u$, hence $p(\hat{u}) + \lambda\hat{u} = p(h^{(+)}) + \lambda h^{(+)}$, and substituting $\lambda = -\bar{p}'(h)$ and using that $p(h^{(+)}) = \bar{p}(h) + \bar{p}'(h)\cdot(h^{(+)} - h)$, one obtains

$$p(\hat{u}) = \bar{p}(h) + \bar{p}'(h)\cdot(\hat{u} - h).$$

Taking into account (3.5), one gets that

$$p(\hat{u}) = \bar{p}(\hat{u}) = \bar{p}(h) + \bar{p}'(h)\cdot(\hat{u} - h),$$

hence $\hat{u} \notin \mathcal{O}_p$. Lemma 4 is proved. $\square$

3.3 Suppose, in addition to the previous assumptions, that there exists the limit $\lim_{u\to+\infty} p(u) =: p(+\infty)$ and that for any $u \in \mathbb{R}_+$, $p(u) > p(+\infty)$. Then $\bar{p}'(u) < 0$ for any u. Denote $B = -\bar{p}'(0)$; one has $B > 0$; the function $\bar{p}'$ is continuous and monotone non-decreasing from B to 0.

Suppose that $d \geq 3$, and introduce an auxiliary notation: $\omega = \frac{1}{d-2}$, $q(u) = |\bar{p}'(u)|^{-\omega}$, $Q(u) = \int_0^u q(\nu)\, d\nu$. Both function q and Q are continuous and monotone non-decreasing on $\mathbb{R}_+$; q changes from $B^{-\omega}$ to $+\infty$, and Q changes from 0 to $+\infty$.

Lemma 5. *Let $d \geq 3$, $h \geq 0$, $p \in C^1(\mathbb{R}_+)$, and let there exist the limit $\lim_{u\to+\infty} p(u) = p(+\infty)$ and for any $u \in \mathbb{R}_+$, $p(u) > p(+\infty)$. Then*

(a) the set of values $U \geq 0$ satisfying the equation

$$U - \frac{Q(U)}{q(U)} = h \tag{3.9}$$

is non-empty.

(b) Let U be a solution of (3.9) and $t_0 = \frac{q(0)}{q(U)}$. Introduce the function $f_h(t)$, $t \in [0, 1]$ as follows:

for $t \in [0, t_0]$, $f_h(t) = -h$;

for $t \in [t_0, 1]$, f_h is defined parametrically,

$$f_h = -h + \frac{u\, q(u) - Q(u)}{q(U)}, \tag{3.10}$$

$$t = \frac{q(u)}{q(U)}, \qquad u \in [0, U]. \tag{3.11}$$

The function f_h is defined correctly, is strictly convex on $[t_0, 1]$, and is a unique solution of the minimization problem (3.2).

(c) The minimal value of $\mathcal{R}$ equals

$$\inf_{f \in \mathcal{M}(h)} \mathcal{R}(f) = \mathcal{R}(f_h) = \bar{p}(U) + \frac{Q(U)}{q(U)^{d-1}} \,. \tag{3.12}$$

Proof. (a) For arbitrary $c > 0$, one has

$$U - \frac{Q(U)}{q(U)} = \int_0^U \left(1 - \frac{q(\nu)}{q(U)}\right) d\nu \geq \int_0^c \left(1 - \frac{q(c)}{q(U)}\right) d\nu,$$

and taking into account that $\lim_{U \to +\infty} q(U) = +\infty$, one concludes that for U sufficiently large, $U - \frac{Q(U)}{q(U)} > c/2$. This implies that the continuous function $U - \frac{Q(U)}{q(U)}$ goes to $+\infty$ as $U \to +\infty$; besides it vanishes at $U = 0$, hence the set of solutions of (3.9) is non-empty. Therefore, (a) is proved.

Denote by $S(h)$ the set of points u such that $\bar{p}(u) = \bar{p}(h) + \bar{p}'(h) \cdot (u - h)$. Note that $S(h)$ coincides with the connected component of $\bar{\mathcal{O}}_p$ containing h, if $h \in \bar{\mathcal{O}}_p$, and $S(h) = \{h\}$ otherwise. Thus, $S(h)$ is a closed segment containing h; two segments $S(h_1)$, $S(h_2)$ either coincide or are disjoint. The condition $p(u) > p(+\infty)$ implies that all segments $S(u)$ are bounded. Obviously, the family of non-degenerated segments (i.e., of those that are not singletons) is at most countable. Denote by $\mathcal{S}$ the union of non-degenerated segments. The function $\bar{p}'(u)$ is monotone increasing on $\mathbb{R}_+ \setminus \mathcal{S}$ and is constant on any non-degenerated segment $S(u) \subset \mathcal{S}$, hence the set $\{\bar{p}'(u),\ u \in \mathcal{S}\}$ is at most countable.

Let $0 \leq u_1 \leq u_2$. After simple algebra one obtains

$$\left[u_2 - \frac{Q(u_2)}{q(u_2)}\right] - \left[u_1 - \frac{Q(u_1)}{q(u_1)}\right] =$$

$$= Q(u_1) \left[\frac{1}{q(u_1)} - \frac{1}{q(u_2)}\right] + \frac{1}{q(u_1)} \int_{u_1}^{u_2} (q(u_2) - q(\nu))\, d\nu. \tag{3.13}$$

Both terms in the right hand side of (3.13) are non-negative, hence the function $U - \frac{Q(U)}{q(U)}$ is monotone non-decreasing. If both u_1 and u_2 are solutions of (3.9) then both terms in (3.13) are equal to zero, which implies that q is constant on $[u_1, u_2]$; that is, $\bar{p}'$ is constant on $[u_1, u_2]$; or, equivalently, $u_1 \in S(u_2)$. This implies that the solution set of (3.9) coincides with a segment $S(u)$.

(b) The relations (3.10) and (3.11) define continuous functions f_h and t of u, varying from $-h$ to 0 and from t_0 to 1, respectively, when u passes the interval $[0, U]$. Moreover, the function $t = q(u)/q(U)$ is monotone non-decreasing, each set $\{u : q(u)/q(U) = t\}$ coincides with some segment $S(u)$, and f_h is constant on any such segment. This means that the function $f_h(t)$ is defined correctly.

Let us calculate the left-hand and right-hand derivatives $f_h'(t^-)$, $f_h'(t^+)$. Designate by $[u^-(t), u^+(t)]$ the interval $\{u : q(u)/q(U) = t\}$ and put $u = u^+(t)$.

Let the values $f_h+\Delta f_h,\ t+\Delta t$ correspond to the argument $u+\Delta u$, and $\Delta t>0$. One has

$$\Delta t = \frac{q(u+\Delta u)-q(u)}{q(U)}\,,$$

$$\Delta f_h = \frac{(u+\Delta u)\,q(u+\Delta u)-Q(u+\Delta u)}{q(U)} - \frac{u\,q(u)-Q(u)}{q(U)} =$$

$$= \frac{u\,(q(u+\Delta u)-q(u)) + \int_u^{u+\Delta u}(q(u+\Delta u)-q(\nu))d\nu}{q(U)}\,,$$

hence

$$\frac{\Delta f_h}{\Delta t} = u + \int_u^{u+\Delta u} \frac{q(u+\Delta u)-q(\nu)}{q(u+\Delta u)-q(u)}\,d\nu\,. \tag{3.14}$$

The integrand in the right hand side of (3.14) is less than 1; due to definition of $u=u^+(t)$, one has $\Delta u\to 0^+$ as $\Delta t\to 0^+$, therefore

$$f_h'(t^+) = \lim_{\Delta t\to 0^+}\frac{\Delta f_h}{\Delta t} = u^+(t).$$

Similarly, one finds

$$f_h'(t^-) = u^-(t).$$

Both functions $u^-(t)$ and $u^+(t)$ are positive, and for any $t_1<t_2$ one has $u^-(t_1)\le u^+(t_1)<u^-(t_2)\le u^+(t_2)$. Therefore, the function f_h is monotone increasing and strictly convex on $[t_0,\,1]$; moreover, it is constant on $[0,\,t_0]$, $f_h(0)=-h$, and $f(1)=0$. Thus, it is proved that $f_h\in\mathcal{M}(h)$.

For any $t\in[t_0,\,1]$, except possibly a countable set of values, one has $u^-(t)=u^+(t):=\tilde u\in\mathbb{R}_+\setminus\mathcal{S}$, hence there exists the derivative $f_h'(t)=\tilde u$. For any $u\ne\tilde u$ one has

$$\bar p(u) > \bar p(\tilde u) + \bar p'(\tilde u)\cdot(u-\tilde u),$$

and using that $p(u)\ge\bar p(u),\ p(\tilde u)=\bar p(\tilde u)$, one obtains

$$p(u) > p(\tilde u) + \bar p'(\tilde u)\cdot(u-\tilde u),$$

hence

$$p(u) - \bar p'(\tilde u)\cdot u > p(\tilde u) - \bar p'(\tilde u)\cdot\tilde u. \tag{3.15}$$

Recall that $t=\frac{q(\tilde u)}{q(U)}=\frac{|\bar p'(U)|^\omega}{|\bar p'(\tilde u)|^\omega}$ and $\omega=\frac{1}{d-2}$. One has $t^{d-2}=\frac{\bar p'(U)}{\bar p'(\tilde u)}$, and multiplying both parts of (3.15) by t^{d-2} and designating $-\bar p'(U)=\lambda$, one obtains that

$$t^{d-2}p(u)+\lambda u > t^{d-2}p(\tilde u)+\lambda\tilde u$$

for any $u\ne\tilde u$. Thus, $\tilde u=f_h'(t)$ is a unique value minimizing the function $t^{d-2}p(u)+\lambda u$.

Let, now, $t\in(0,\,t_0)$. For $u>0$ one has

$$\bar p(u)\ge\bar p(0)+\bar p'(0)\cdot u.$$

Using that $p(u) \geq \bar{p}(u)$, $p(0) = \bar{p}(0)$, $t_0^{2-d} = \bar{p}'(0)/\bar{p}'(U) = -\bar{p}'(0)/\lambda$, one obtains

$$p(u) \geq p(0) + \bar{p}'(0) \cdot u = p(0) - \lambda t_0^{2-d} u,$$

hence for any $t \in (0,\, t_0)$

$$p(u) + \lambda t^{2-d} u > p(0),$$

therefore the value $f_h'(t) = 0$ is a unique minimum of the function $t^{d-2} p(u) + \lambda u$. Applying lemma 3, one concludes that f_h is a unique solution of (3.2).

(c) One has

$$\mathcal{R}(f_h) = \int_0^{t_0} p(0)\, dt^{d-1} + \int_{t_0}^1 p(f_h'(t))\, dt^{d-1}. \tag{3.16}$$

The first integral in the right hand side of (3.16) equals

$$\int_0^{t_0} \ldots = p(0) \left(\frac{q(0)}{q(U)} \right)^{d-1}.$$

Denote $\tilde{U} = \inf S(U)$. Changing the variable in the second integral $t = q(u)/q(U)$, $u \in [0,\, \tilde{U}]$ and taking into account that for almost every t, $f_h'(t) = u$, one obtains that the second integral equals

$$\int_{t_0}^1 \ldots = \int_0^{\tilde{U}} p(u)\, d\!\left(\frac{q(u)}{q(U)} \right)^{d-1} = p(u) \left(\frac{q(u)}{q(U)} \right)^{d-1} \Bigg|_0^{\tilde{U}} - \int_0^{\tilde{U}} \left(\frac{q(u)}{q(U)} \right)^{d-1} dp(u).$$

Summing the first and the second integrals and taking into account that $q(\tilde{U}) = q(U)$ and $d - 1 = 1 + 1/\omega$, one obtains

$$\mathcal{R}(f_h) = p(\tilde{U}) - \int_0^{\tilde{U}} \left(\frac{q(u)}{q(U)} \right)^{1+1/\omega} dp(u). \tag{3.17}$$

The integral in (3.17) can be represented as the sum

$$\int_0^{\tilde{U}} (\ldots) = \int_{[0,\tilde{U}] \setminus \mathcal{S}} (\ldots) + \sum_i \int_{S_i} (\ldots),$$

where $S_i = S(u_i)$ are non-degenerated segments whose union gives $[0,\, \tilde{U}] \cap \mathcal{S}$. If $u \in [0,\, \tilde{U}] \setminus \mathcal{S}$, one has $p'(u) = \bar{p}'(u) = -q(u)^{-1/\omega}$, hence

$$\int_{[0,\tilde{U}] \setminus \mathcal{S}} (\ldots) = -\int_{[0,\tilde{U}] \setminus \mathcal{S}} \frac{q(u)}{q(U)^{1+1/\omega}}\, du.$$

Next, taking into account that the function q is constant on S_i and that at endpoints of S_i, p and $\bar{p}$ coincide, one obtains

$$\int_{S_i} (\ldots) = \int_{S_i} \left(\frac{q(u)}{q(U)} \right)^{1+1/\omega} d\bar{u} = -\int_{S_i} \frac{q(u)}{q(U)^{1+1/\omega}}\, du.$$

Summing these integrals, one gets

$$\int_0^{\tilde{U}} \left(\frac{q(u)}{q(U)}\right)^{1+1/\omega} dp(u) \;=\; -\int_0^{\tilde{U}} \frac{q(u)}{q(U)^{1+1/\omega}}\, du. \tag{3.18}$$

Further, the function $q(u) = |\bar{p}'(u)|^{-\omega}$ is constant on $[\tilde{U},\, U]$, hence

$$-\int_{\tilde{U}}^{U} \frac{q(u)}{q(U)^{1+1/\omega}}\, du = -q(U)^{-1/\omega}(U-\tilde{U}) = \bar{p}'(U)\,(U-\tilde{U}) = \bar{p}(U) - \bar{p}(\tilde{U}). \tag{3.19}$$

Using that $\bar{p}(\tilde{U}) = p(\tilde{U})$ and applying (3.17), (3.18), and (3.19), one gets

$$\mathcal{R}(f_h) = \bar{p}(U) + \int_0^{U} \frac{q(u)}{q(U)^{1+1/\omega}}\, du,$$

and recalling that Q is the primitive of q and $1 + 1/\omega = d - 1$, one comes to the formula (3.12). $\square$

4 Solution of the minimal resistance problem

4.1 Two-dimensional problem

4.1.1 Minimization of $\mathcal{R}_+$

From statement (a) of lemma 2 it follows that there exist values $u_+^0 > 0$ and $B_+ > 0$ such that

$$\frac{p_+(u_+^0) - p_+(0)}{u_+^0} = p'_+(u_+^0) = -B_+$$

and

$$\bar{p}_+(u) = \begin{cases} p_+(0) - B_+ u & \text{if } 0 \le u \le u_+^0, \\ p_+(u) & \text{if } u \ge u_+^0. \end{cases}$$

This implies that $\mathcal{O}_{p_+} = (0,\, u_+^0)$. Applying lemma 4, one obtains that there exists a unique solution f_h^+ of the minimization problem

$$\inf_{f \in \mathcal{M}(h)} \mathcal{R}_+(f), \qquad \mathcal{R}_+(f) = \int_0^1 p_+(f'(t))\, dt,$$

defined by the relations

$$\begin{aligned} f_h^+(t) \;&=\; \begin{cases} -h & \text{for } t \in [0,\; t_0], \\ -h + u_+^0 \cdot (t - t_0) & \text{for } t \in [t_0,\, 1], \end{cases} \\ t_0 \;&=\; 1 - h/u_+^0, \end{aligned} \tag{4.1}$$

if $0 \le h < u_+^0$, and

$$f_h^+(t) = -h + ht,$$

if $h \ge u_+^0$. The minimal resistance equals

$$\inf_{f \in \mathcal{M}(h)} \mathcal{R}_+(f) = \mathcal{R}_+(f_h^+) = \bar{p}_+(h).$$

4.1.2 Minimization of $\mathcal{R}_-$

Note that $p'_-(0) = 0$ and $\bar{p}'_-(0) < 0$, hence $\mathcal{O}_{p_-}$ contains an interval $(0,\, u_-^0)$, $u_-^0 > 0$, besides $p_-(u_-^0) = p_-(0) + \bar{p}'_-(0) \cdot u_-^0$. Denote $B_- = -\bar{p}'_-(0)$ and represent the open set $\mathcal{O}_{p_-}$ as the union of its connected components $\mathcal{O}_i = (u_i^-,\, u_i^+)$, $\mathcal{O}_{p_-} = \cup_i \mathcal{O}_i$. We shall suppose that the set of indices $\{i\}$ contains 1 and that $\mathcal{O}_1 = (0,\, u_-^0)$. Statement (b) of lemma 2 and example 2 imply that in some cases (for example, when considering pressure distribution of a mixture of two homogeneous rarefied gases on the rear part of surface of a moving body) $\mathcal{O}_{p_-}$ has at least two connected components.

Consider the minimization problem

$$\inf_{f \in \mathcal{M}(h)} \mathcal{R}_-(f), \qquad \mathcal{R}_-(f) = \int_0^1 p_-(f'(t))\, dt. \tag{4.2}$$

Applying lemma 4, one obtains that there exists a solution f_h^- of this problem, besides

$$\inf_{f \in \mathcal{M}(h)} \mathcal{R}_-(f) = \mathcal{R}_-(f_h^-) = \bar{p}_-(h).$$

For $0 \le h < u_-^0$ one has

$$\begin{aligned} f_h^-(t) &= \begin{cases} -h, & \text{if } t \le t_0 \\ -h + u_-^0 \cdot (t - t_0), & \text{if } t \ge t_0, \end{cases} \\ t_0 &= 1 - h/u_-^0. \end{aligned} \tag{4.3}$$

For $h \in \mathbb{R} \setminus \mathcal{O}_{p_-}$ one has

$$f_h^-(t) = -h + ht.$$

Finally, for $h \in \mathcal{O}_i,\ i \neq 1$ one has

$$\begin{aligned} f_h^-(t) &= \begin{cases} -h + u_i^- t, & \text{if } t \le t_i \\ -h + u_i^- t_i + u_i^+(t - t_i), & \text{if } t \ge t_i\,, \end{cases} \\ t_i &= \frac{u_i^+ - h}{u_i^+ - u_i^-}. \end{aligned} \tag{4.4}$$

Notice that f_h needs not be the unique solution of (4.2). In some degenerated cases it may happen that the right endpoint of some interval $\mathcal{O}_i$ coincides with the left endpoint of another interval, $u_i^+ = u_j^-,\ i \neq j;$ then there exists a continuous family of solutions of (4.2); the derivative of any function from this family takes the values $u_i^-,\ u_i^+$, and u_j^+.

4.1.3 Solution of the two-dimensional problem

Thus, the problem of finding

$$\mathrm{R}(h) = \inf_{h_+ + h_- = h} \left(\mathcal{R}_+(f_{h_+}^+) + \mathcal{R}_-(f_{h_-}^-) \right)$$

amounts to the problem

$$\min_{0\le z\le h} p_h(z), \quad \text{where} \quad p_h(z) = \bar p_+(z) + \bar p_-(h-z). \tag{4.5}$$

The functions $\bar p'_-(u)$, $\bar p'_+(u)$ are continuous and monotone non-decreasing, hence the function $p'_h(z)$, $0 \le z \le h$ is also continuous and monotone non-decreasing.

Using statement (c) of lemma 1, one concludes that $B_+ > B_-$. Indeed, if $u^0_- \le u^0_+$ then

$$-B_- = \frac{p_-(u^0_-) - p_-(0)}{u^0_-} > \frac{p_+(u^0_-) - p_+(0)}{u^0_-} \ge \frac{\bar p_+(u^0_-) - p_+(0)}{u^0_-} = -B_+,$$

and if $u^0_- > u^0_+$ then

$$-B_- = p'_-(u^0_-) > p'_+(u^0_-) > p'_+(u^0_+) = -B_+.$$

Thus, there exists a unique value $u_* > u^0_+$ such that $\bar p'_+(u_*) = p'_+(u_*) = -B_-$. Consider four cases:

1) $0 < h < u^0_+$;
2) $u^0_+ \le h \le u_*$;
3) $u_* < h < u_* + u^0_-$;
4) $h \ge u_* + u^0_-$.

In the cases 1) and 2), for $0 \le z < h$, one has $p'_h(z) < \bar p'_+(u_*) + B_- = 0$, hence $z = h$ is a unique value of argument minimizing p_h. Therefore, the optimal values of h_+ and h_- are $h_+ = h$, $h_- = 0$, and $f^-_{h_-=0} \equiv 0$.

1) $0 < h < u^0_+$. The function $f^+_{h_+=h}$ is given by (4.1). The body of least resistance is a trapezium, the tangent of slope of its lateral sides being equal to u^0_+ (see Fig. 2(a)). The minimal resistance equals

$$\mathrm{R}(h) = \mathcal{R}_+(f^+_{h_+}) + \mathcal{R}_-(f^-_{h_-}) = p_+(0) - B_+\, h + p_-(0).$$

2) $u^0_+ \le h \le u_*$. Here one has $f^+_{h_+=h}(t) = -h + h\,t$, hence the optimal body is an isosceles triangle (see Fig. 2(b)), and

$$\mathrm{R}(h) = p_+(h) + p_-(0).$$

In the cases 3) and 4) one has $\bar p'_+(h) > -B_-$, hence $p'_h(h) = \bar p'_+(h) - \bar p'_-(0) > 0$. On the other hand, $p'_h(u^0_+) = \bar p'_+(u^0_+) - \bar p'_-(h - u^0_+) \le -B_+ + B_- < 0$. Moreover, using statement (a) of lemma 2, one finds that the function $\bar p'_+(u) = p'_+(u)$, $u \in [u^0_+,\, h]$ is monotone increasing, hence p'_h is also monotone increasing on this interval. It follows that the function p_h has a unique minimum $z \in (u^0_+,\, h)$ and $f^+_{h_+=z}(t) = -z + zt$.

3) $u_* < h < u_* + u^0_-$. One has $p'_h(u_*) = \bar p'_+(u_*) - \bar p'_-(h - u_*) = -B_- + B_- = 0$, therefore p_h reaches its minimal value at $z = u_*$; thus, the optimal values of h_+ and h_- are $h_+ = u_*$, $h_- = h - u_*$. The function $f^-_{h_-=h-u_*}$ is given by (4.3). Here the optimal body is the union of a triangle and a trapezium, as shown on

Fig. 2(c). The tangent of slope of lateral sides of the trapezium equals $-u^0_-$. The minimal resistance equals

$$\mathrm{R}(h) = p_+(u_*) + p_-(0) - B_-\,(h - u_*).$$

4) $h \ge u_* + u^0_-$. One has $p'_h(h - u^0_-) = \bar{p}'_+(h - u^0_-) + B_- \ge 0,$ hence the minimum of p_h is reached at a point $z \in (u^0_+,\, h - u^0_-]$, and the optimal values $h_+ = z,\ h_- = h - z$, as well as the minimal resistance, are obtained from the relations

$$\begin{array}{l} h_+ + h_- = h, \\ p'_+(h_+) = p'_-(h_-), \\ h_+ \ge u^0_+,\ h_- \ge u^0_-, \\ \mathrm{R}(h) = p_+(h_+) + \bar{p}_-(h_-). \end{array}$$

Here one should distinguish between two cases.

4a) If $h_- \in \mathbb{R} \setminus \mathcal{O}_{p_-}$ then $f^-_{h_-}(t) = -h_- + h_- t$, and the optimal body is a union of two isosceles triangles with common base, of heights h_+ and h_- (see Fig. 2(d)).

4b) If h_- belongs to some interval $\mathcal{O}_i = (u^-_i,\, u^+_i),\ i \ne 1$, then $f^-_{h_-}$ is given by (4.4), and the optimal body is the union of two isosceles triangles and a trapezium (see Fig. 2(e)).

Note that the case 4b) is realized for the values h from an open (maybe empty) set contained in $(u_* + u^0_-,\, +\infty)$. This set is defined by the parameters σ and V. The case 4a) is realized for the values h from the complement of this set in $(u_* + u^0_-,\, +\infty)$, which is always non-empty.

4.2 The problem in three and more dimensions

Let $d \ge 3$. Using lemma 5, one concludes that there exists a unique solution f^ε_h of the problem

$$\inf_{f \in \mathcal{M}(h)} \mathcal{R}_\varepsilon(f), \qquad \mathcal{R}_\varepsilon(f) = \int_0^1 p_\varepsilon(f'(t))\, dt^{d-1},$$

besides

$$\mathcal{R}_\varepsilon(f^\varepsilon_h) = \bar{p}_\varepsilon(U) + \frac{Q_\varepsilon(U)}{q_\varepsilon(U)^{d-1}}\,,$$

where U is defined (not necessarily uniquely) by the relation

$$U - \frac{Q_\varepsilon(U)}{q_\varepsilon(U)} = h;$$

here $q_\varepsilon(U) = |\bar{p}'_\varepsilon(U)|^{-1/(d-2)}$, $Q_\varepsilon(U) = \int_0^U q_\varepsilon(u)\, du$.

Thus, the problem

$$\inf_{h_+ + h_- = h} \Big(\mathcal{R}_+(f^+_{h_+}) + \mathcal{R}_-(f^-_{h_-}) \Big)$$

amounts to the following problem:

$$\inf_{\mathrm{h}_+(u_+)+\mathrm{h}_-(u_-)=h} \left(\mathrm{r}_+(u_+) + \mathrm{r}_-(u_-)\right),$$

where

$$\mathrm{r}_+(u) = \bar{p}_+(u) + \frac{Q_+(u)}{q_+(u)^{d-1}}, \qquad \mathrm{r}_-(u) = \bar{p}_-(u) + \frac{Q_-(u)}{q_-(u)^{d-1}}$$

and

$$\mathrm{h}_+(u) = u - \frac{Q_+(u)}{q_+(u)}, \qquad \mathrm{h}_-(u) = u - \frac{Q_-(u)}{q_-(u)}, \quad u \geq 0.$$

The functions r_ε and $\bar{p}_\varepsilon$, $\varepsilon \in \{-, +\}$ are monotone non-increasing, and h_ε is monotone non-decreasing from 0 to $+\infty$ when $u \in \mathbb{R}_+$, besides any interval of constancy of one of these functions is at the same time the interval of constancy of two others. For each $z \geq 0$ choose u such that $\mathrm{h}_\varepsilon(u) = z$ and put $\mathrm{r}^{(\varepsilon)}(z) := \mathrm{r}_\varepsilon(u)$, $\pi^{(\varepsilon)}(z) := \bar{p}'_\varepsilon(u)$. From the stated above it follows that the functions $\mathrm{r}^{(\varepsilon)}$ and $\pi^{(\varepsilon)}$ well defined on $\mathbb{R}_+$ and are monotone decreasing. Denote

$$\mathrm{r}_h(z) = \mathrm{r}^{(+)}(z) + \mathrm{r}^{(-)}(h - z).$$

After some algebra one obtains that the function r_h is differentiable and

$$\mathrm{r}'_h(z) = (d-1)\,(\bar{p}'_+(u_+) - \bar{p}'_-(u_-)), \tag{4.6}$$

where the values $u_+,\ u_-$ are chosen from the relations $\mathrm{h}_+(u_+) = z,\ \mathrm{h}_-(u_-) = h - z$. Both values in the right hand side of (4.6), $\bar{p}'_+(u_+) = \pi^{(+)}(z)$ and $-\bar{p}'_-(u_-) = -\pi^{(-)}(h - z)$, are monotone increasing functions of z, hence $\mathrm{r}'_h(z)$ is also monotone increasing from $\mathrm{r}'_h(0) = (d-1)\,(\bar{p}'_+(0) - \bar{p}'_-(U_-))$ to $\mathrm{r}'_h(h) = (d-1)\,(\bar{p}'_+(U_+) - \bar{p}'_-(0))$, where U_+ and U_- are defined from the relations $\mathrm{h}_+(U_+) = h,\ \mathrm{h}_-(U_-) = h$. Note that $\bar{p}'_+(0) = -B_+$ and $\bar{p}'_-(U_-) \geq -B_- > -B_+$, therefore $\mathrm{r}'_h(0) < 0$.

Recall that u_* is defined in the subsection 4.1.3 by $\bar{p}'_+(u_*) = -B_-$. Designate

$$h_* := \mathrm{h}_+(u_*) = u_* - B_-^{\frac{1}{d-2}}\, Q_+(u_*) \tag{4.7}$$

and consider two cases.

1) $h \leq h_*$. One has $\mathrm{h}_+(U_+) \leq \mathrm{h}_+(u_*)$, hence $U_+ \leq u_*$, therefore $\mathrm{r}'_h(h) = (d-1)\,(\bar{p}'_+(U_+) + B_-) \leq (d-1)\,(\bar{p}'_+(u_*) + B_-) = 0$. This implies that $\mathrm{r}'_h(z) < 0$ for $z \in [0,\, h)$, hence the function r_h has a unique minimum at the point $z = h$, which corresponds to the values $h_+ = h,\ h_- = 0$. The minimal resistance equals

$$\mathrm{R}(h) = \bar{p}_+(u_+) + Q_+(u_+)\, q_+(u_+)^{-d+1} + \bar{p}_-(0).$$

2) $h > h_*$. One has $U_+ > u_*$, therefore $\mathrm{r}'_h(h) = (d-1)\,(\bar{p}'_+(U_+) + B_-) > 0$. On the other hand, $\mathrm{r}'_h(0) < 0$. Hence, there exists a unique value $z \in (0,\, h)$ such that $\mathrm{r}'_h(z) = 0$. Thus, the function r_h has a unique minimum at z; the optimal

values of $h_+,\ h_-$ are $h_+ = z > 0,\ h_- = h - z > 0$. These values and the related auxiliary values $u_-,\ u_+$ are uniquely defined from the system of four equations

$$
\begin{aligned}
&h_+ = u_+ - Q_+(u_+)/q_+(u_+),\\
&h_- = u_- - Q_-(u_-)/q_-(u_-),\\
&h_+ + h_- = h,\\
&\bar{p}'_+(u_+) = \bar{p}'_-(u_-),
\end{aligned}
$$

and the minimal resistance equals

$$\mathrm{R}(h) = \bar{p}_+(u_+) + Q_+(u_+)\, q_+(u_+)^{-d+1} + \bar{p}_-(u_-) + Q_-(u_-)\, q_-(u_-)^{-d+1}.$$

4.3 The limit cases

Consider heuristically the limit behavior of solutions as $V \to +\infty$ and as $V \to 0$, with fixed h and σ. We shall denote the pressure and resistance functions by $p_\pm(u, V)$ and $\mathrm{R}(h, V)$, thus explicitly indicating dependence of these functions on the parameter V.

4.3.1 $\mathrm{V} \to +\infty$

Denote by $\tilde{p}_\varepsilon(u, V) = V^{-2} p_\varepsilon(u, V),\ \varepsilon \in \{-, +\}$ the reduced pressure, and by $\tilde{\mathrm{R}}(h, V) = V^{-2}\mathrm{R}(h, V)$, the minimal reduced resistance. One has

$$
\begin{aligned}
&\tilde{p}_+(u, V) = 1/(1 + u^2) + o(1),\\
&\tilde{p}_-(u, V) = o(1), \quad \tilde{p}'_-(u, V) = o(1), \quad V \to +\infty;
\end{aligned}
$$

in other words, as $V \to +\infty$, the functions $\tilde{p}_+(u, V)$ and $\tilde{p}_-(u, V)$ tend to $1/(1 + u^2)$ and to 0, respectively. These limit functions determine pressure distribution on the front part and on the rear part of body's surface in Newton's classical problem.

Consider the cases $d = 2$ and $d = 3$ separately.

$\mathbf{d = 2}$ If $h < 1$ then for V sufficiently large, the figure of least resistance is a trapezium, and the inclination angle of its lateral sides tends to 45^0 as $V \to +\infty$. If $h > 1$ then for V sufficiently large, the figure of least resistance is an isosceles triangle coincident with the solution of two-dimensional Newton's problem.

$\mathbf{d = 3}$ For V sufficiently large, the body of least resistance is the first kind solution. The front part of its surface is the the graph of a function defined on a unit circle; as $V \to +\infty$, this function uniformly converges to the function that describes Newton's classical solution with the same h.

The case $d > 3$ is similar to the three-dimensional one.

Finally, the limit value of minimal reduced resistance $\tilde{\mathrm{R}}(h, \infty)$ is equal to the resistance of Newton's optimal solution multiplied by the density of particles' flow $\nu = \int_{\mathbb{R}^d} \sigma(|v|)\, dv$.

4.3.2 $V \to 0^+$

In this limit case one has

$$p_\varepsilon(u, V) = \varepsilon b^{(d)} + V \frac{c^{(d)}}{\sqrt{1+u^2}} + o(V), \qquad \varepsilon \in \{-, +\},$$

where

$$b^{(2)} = \frac{\pi}{2} \int_0^{+\infty} \sigma(r) r^3\, dr, \qquad c^{(2)} = 4 \int_0^{+\infty} \sigma(r) r^2\, dr, \tag{4.8}$$

$$b^{(3)} = \frac{2\pi}{3} \int_0^{+\infty} \sigma(r) r^4\, dr, \qquad c^{(3)} = 2\pi \int_0^{+\infty} \sigma(r) r^3\, dr. \tag{4.9}$$

This formula will be derived in Appendix B. One readily obtains that u_-^0 and u_+^0 tend to the value $a := \sqrt{(1+\sqrt{5})/2} \approx 1.272$, and $B_\pm = V \cdot a^{-5} + o(V)$. Taking into account that $\bar{p}'_+(u) < \bar{p}'_-(u) < 0$, one concludes that u_* tends to the same value a, and $u_+^0 + u_*$ tends to $2a$.

Let us describe the shape of optimal body and determine the minimal reduced resistance $\hat{\mathrm{R}}(h, V) = V^{-1}\mathrm{R}(h, V)$ in the limit $V \to 0^+$. We shall distinguish between two cases: $d = 2$ and $d = 3$.

d = 2 (a) $0 < h < a$: the optimal body is a trapezium.
(b) $h = a$: an isosceles triangle.
(c) $a < h < 2a$: the union of a triangle and a trapezium.
(d) $h \geq 2a$: a rhombus.

In the first three cases, the tangent of slope of lateral sides of optimal figures equals $\arctan a \approx 51.8^0$, and in the last case, exceeds this value. The examples of optimal figures are shown on Fig. 4.

The minimal reduced resistance equals

$$\hat{\mathrm{R}}(h, V) = 2c^{(2)} \bar{p}(h/2) + o(1), \qquad V \to 0^+, \tag{4.10}$$

where

$$\bar{p}(u) = \begin{cases} 1 - a^{-5}\, u, & \text{if } u \leq a, \\ 1/\sqrt{1+u^2}, & \text{if } u \geq a. \end{cases} \tag{4.11}$$

d = 3 Let

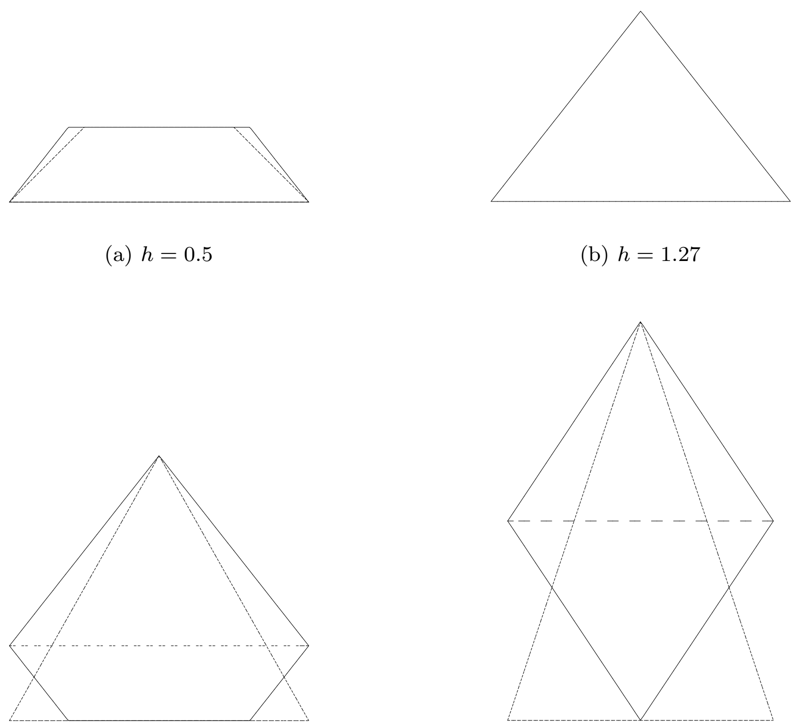
$$q(u) = \begin{cases} a^5 & \text{if } u \leq a \\ \frac{(1+u^2)^{3/2}}{u} & \text{if } u \geq a, \end{cases}$$

$$Q(u) = \begin{cases} a^5 u & \text{if } u \leq a \\ \sqrt{1+u^2}\, \frac{4+u^2}{3} + \frac{2+a^2}{3} + \ln \frac{\sqrt{1+u^2}-1}{u} - \ln \frac{a^2-1}{a} & \text{if } u \geq a, \end{cases}$$

and let U be a (unique) solution of (3.9). Define the function f_h like in the statement (b) of lemma 5. Then the body of least resistance is

$$\{(x', x_3) \in \mathbb{R}^3 : |x'| \leq 1,\ |x_3| \leq -f_h(|x'|)\},$$

where $x' = (x_1, x_2)$. Thus, the body is symmetric with respect to the horizontal plane $\{x_3 = 0\}$. The front and rear parts of its surface contain flat circular disks



(a) $h = 0.5$

(b) $h = 1.27$

(c) $h = 1.77$

(d) $h = 3$

Figure 4: Two-dimensional case. Solutions in the limit $V \to 0^+$ are shown by solid line. The corresponding solutions of Newton's classical problem are shown by dashed line. The case (b) is the unique one where these two solutions coincide.

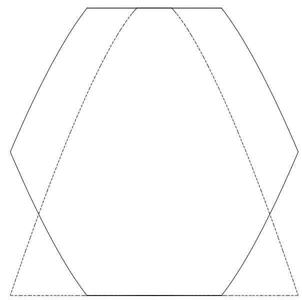


Figure 5: Three-dimensional case, $h = 1$. The solution in the limit $V \to 0^+$ (solid line) and the solution of Newton's classical problem (dashed line).

of equal radius, and the angle of slope of lateral surface near these disks equals $\arctan a \approx 51.8^0$.

On Fig. 5, the projections of two optimal bodies of height $h = 1$ on the plane Ox_1x_3 are shown. The bodies are: the body of least resistance in the limit $V \to 0^+$ and the solution of Newton's classical problem.

The minimal reduced resistance equals

$$\hat{\mathrm{R}}(h, V) = 2c^{(3)} \left(\bar{p}(U) + \frac{Q(U)}{q^2(U)} \right),$$

where $\bar{p}$ is defined in (4.11), and U is given by (3.9).

It is interesting to note that in these limit cases, the shape of optimal body does not depend on the distribution σ; moreover, the reduced minimal resistance is proportional to ν in the limit $V \to +\infty$, and is proportional to the factor $c^{(d)}$ given by (4.8), (4.9) in the limit $V \to 0^+$. This factor can be interpreted as the sum of absolute values of impulses of particles of the medium in unit volume, in the frame of reference associated with the medium.

5 Gaussian distribution of velocities: exact solutions

Suppose that the function $\rho = \rho_V$ is the density of circular gaussian distribution with mean $-Ve_d$ and variance 1, i.e.,

$$\rho_V(v) = \sigma(|v + Ve_d|), \quad \text{where} \quad \sigma(r) = (2\pi)^{-d/2}\, e^{-r^2/2}. \tag{5.1}$$

This function describes the particles' distribution over velocities in a frame of reference moving in a homogeneous monatomic ideal gas, where the velocity of motion equals V and the mean square velocity of molecules equals 1 (see example 1).

The function ρ_V satisfies the condition A, therefore the results obtained in the previous section can be applied in this case. Here, the pressure functions

$p_\pm(u,V)$ are calculated analytically in the cases $d = 2$ and $d = 3$, and then, using numerical simulation, the following results are obtained:

1) The parameter set V-h is divided into several subsets corresponding to different kinds of solutions. This partition is shown on Figures 6 and 9.

2) The least resistance $\mathrm{R}(h,V)$ is calculated for various values of h, V. The results are shown on Figures 7 and 10.

3) For several values of parameters h and V, the body of least resistance is constructed. Two such bodies are shown on Figures 1, 2 (a)–(d) and 8.

Here the value V is allowed to vary, so the pressure and resistance functions are designated by $p_\pm(u,V)$ and $\mathrm{R}(h,V)$ instead of $p_\pm(u)$ and $\mathrm{R}(h)$.

Consider the cases $d = 2$ and $d = 3$.

5.1 Two-dimensional case

Fixing the sign "+" and passing to polar coordinates $v = (-r\sin\varphi, -r\cos\varphi)$ in the formula (2.10), one obtains

$$p_+(u,V) = \iint \frac{r^2(\cos\varphi + u\sin\varphi)_+{}^2}{1+u^2}\,\rho_+(r,\varphi,V)\,rdrd\varphi, \tag{5.2}$$

where $z_+ := \max\{0,\, z\}$, and $\rho_+(r,\varphi,V)$ is the gaussian density function ρ_V (5.1) written in the introduced polar coordinates,

$$\rho_+(r,\varphi,V) = \frac{1}{2\pi}\,e^{-\frac{1}{2}(r^2 - 2Vr\cos\varphi + V^2)}. \tag{5.3}$$

Next, fixing the sign "−" and introducing polar coordinates in a slightly different manner, $v = (-r\sin\varphi, r\cos\varphi)$, one obtains

$$p_-(u,V) = -\iint \frac{r^2(\cos\varphi + u\sin\varphi)_+{}^2}{1+u^2}\,\rho_-(r,\varphi,V)\,rdrd\varphi, \tag{5.4}$$

Here $\rho_-(r,\varphi,V)$ is the same density function ρ_V (5.1) written in these coordinates,

$$\rho_-(r,\varphi,V) = \frac{1}{2\pi}\,e^{-\frac{1}{2}(r^2 + 2rV\cos\varphi + V^2)}. \tag{5.5}$$

Combining the formulas (5.2), (5.3), (5.4), and (5.5), one comes to the more general expression

$$p_\varepsilon(u,V) = \varepsilon\,\frac{e^{-V^2/2}}{2\pi} \iint\limits_{\cos\varphi + u\sin\varphi > 0} \frac{(\cos\varphi + u\sin\varphi)^2}{1+u^2}\; e^{-\frac{1}{2}r^2 + \varepsilon 2rV\cos\varphi}\; r^3\,drd\varphi\,,$$

where $\varepsilon \in \{-,+\}$. Passing to the iterated integral and integrating over r, one obtains

$$p_\varepsilon(u,V) = \varepsilon\,\frac{e^{-V^2/2}}{\pi} \int\limits_{\cos\varphi + u\sin\varphi > 0} \frac{(\cos\varphi + u\sin\varphi)^2}{1+u^2}\; l(\varepsilon V\cos\varphi)\,d\varphi\,, \tag{5.6}$$

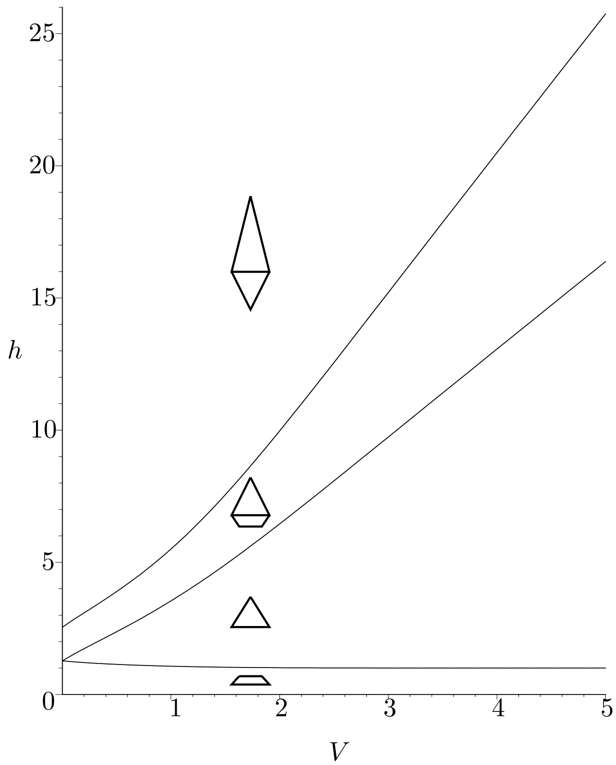


Figure 6: Two-dimensional case. Four regions shown on the parameter space correspond to four kinds of solutions.

where

$$l(z) = 1 + \frac{z^2}{2} + \frac{\sqrt{\pi}}{2\sqrt{2}}\, e^{z^2/2} \left(3z + z^3\right) \left(1 + \operatorname{erf}\left(z/\sqrt{2}\right)\right),$$

and $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. Changing the variable $\tau = \varphi - \arcsin(u/\sqrt{1+u^2})$, one finally comes to

$$p_\varepsilon(u, V) = \varepsilon \frac{e^{-V^2/2}}{\pi} \int_{-\pi/2}^{\pi/2} \cos^2 \tau \; l\left(\varepsilon V \, \frac{\cos\tau - u \sin\tau}{\sqrt{1+u^2}}\right) d\tau. \tag{5.7}$$

Numerical simulations are done using Maple and verified by Matlab.

Graphs of the functions

$$h = u_+^0(V), \quad h = u_*(V), \quad h = u_*(V) + u_-^0(V),$$

are shown on Figure 6, where the values u_+^0, u_-^0 and u_* (which are the functions of V) are defined in subsections 4.1.1, 4.1.2 and 4.1.3, respectively. These graphs separate the parameter space $\mathbb{R}_+^2$ into four regions corresponding to the four

different kinds of solutions. The lower function $h = u_+^0(V)$ tends to 1 as $V \to \infty$. Further, at $V = 0$, the lower, the middle, and the upper functions take the values $a,\ a,$ and $2a$, respectively, where $a = \sqrt{(1+\sqrt{5})/2} \approx 1.272$: $\lim_{V\to 0} u_+^0(V) = a,\ \lim_{V\to 0} u_*(V) = a,\ \lim_{V\to 0}(u_*(V) + u_-^0(V)) = 2a$.

The solutions of fifth kind (union of two triangles and a trapezium) were not found in numerical simulations. (These solutions correspond to the case where the set $\mathcal{O}_{p_-,V} = \{u : \bar p_-(u,V) < p_-(u,V)\}$ contains at least two connected components.) We believe that in the considered case corresponding to the gaussian distribution ρ_V this kind of solutions does not appear at all. (Notice that, according to the statement (b) of lemma 2, this kind of solutions does appear for some distributions corresponding to mixtures of homogeneous gases.)

Further, using the formulas from subsections 4.1.1 and 4.1.2, the functions f_+ and f_- are calculated, which allow one to construct the optimal figures (Fig. 2 (a)–(d)), and using the formulas from 4.1.3, the minimal resistance $\mathrm{R}(h,V)$ is calculated. The graphs of reduced minimal resistance $\tilde{\mathrm{R}}(h,V) = V^{-2}\mathrm{R}(h,V)$ versus h are shown on figures 7(a) and 7(b) for several values of V.

5.2 Three-dimensional case

Fix the sign "+". In spherical coordinates $v = (-r\sin\varphi\cos\theta,\ -r\sin\varphi\sin\theta,\ -r\cos\varphi)$, $r \ge 0,\ 0 \le \varphi \le \pi,\ -\pi \le \theta \le \pi$ the formula (2.10) takes the form

$$p_+(u,V) = \iiint \frac{r^2(\cos\varphi + u\sin\varphi\cos\theta)_+{}^2}{1+u^2}\, \rho_+(r,\varphi,\theta,V)\, r^2 \sin\varphi\, dr d\varphi d\theta,$$

where

$$\rho_+(r,\varphi,\theta,V) = \frac{1}{(2\pi)^{3/2}}\, e^{-\frac{1}{2}(r^2 - 2Vr\cos\varphi + V^2)}.$$

Now, fix the sign "−". In spherical coordinates $v = (-r\sin\varphi\cos\theta,\ -r\sin\varphi\sin\theta,\ r\cos\varphi)$, one has

$$p_-(u,V) = -\iiint \frac{r^2(\cos\varphi + u\sin\varphi\cos\theta)_+{}^2}{1+u^2}\, \rho_-(r,\varphi,\theta,V)\, r^2 \sin\varphi\, dr d\varphi d\theta,$$

where

$$\rho_-(r,\varphi,\theta,V) = \frac{1}{(2\pi)^{3/2}}\, e^{-\frac{1}{2}(r^2 + 2Vr\cos\varphi + V^2)}.$$

Summarizing, one comes to the formula

$$p_\varepsilon(u,V) = \varepsilon\, \frac{e^{-V^2/2}}{(2\pi)^{3/2}} \iiint_{\cos\varphi + u\sin\varphi\cos\theta > 0} \frac{(\cos\varphi + u\sin\varphi\ cos\theta)^2}{1+u^2}\ \cdot$$

$$\cdot\, e^{-\frac{1}{2}r^2 + \varepsilon Vr\cos\varphi} r^4 \sin\varphi\, dr d\varphi d\theta, \qquad \varepsilon \in \{-,+\}. \tag{5.8}$$

We shall use two formulas, which are easy to verify. First,

$$\int_0^{+\infty} e^{-\frac{1}{2}r^2 + \varepsilon Vr\cos\varphi} r^4\, dr = I(\varepsilon V\cos\varphi),$$

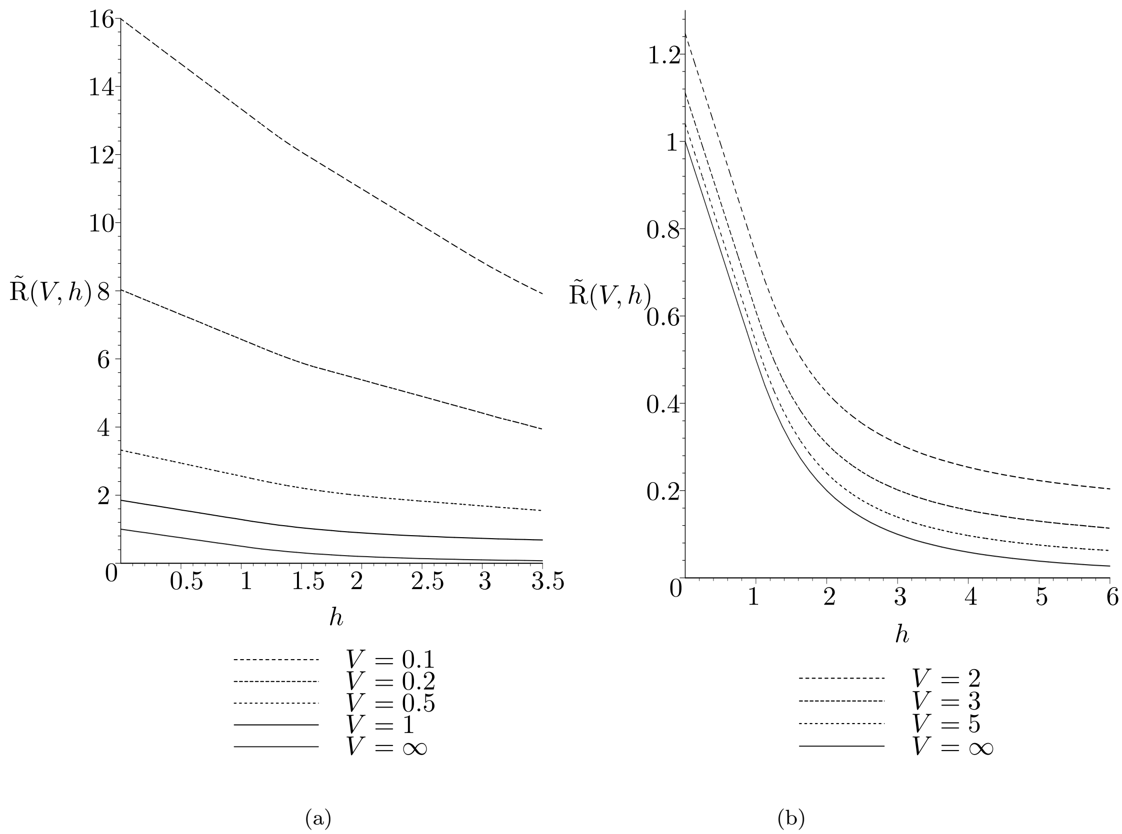


Figure 7: Two-dimensional case. Least reduced resistance $\tilde{\mathrm{R}}(V,h)$ versus height h of the body

where

$$I(z) = \sqrt{\pi/2}\, e^{z^2/2}\,(3+6z^2+z^4)(1+\mathrm{erf}\,(z/\sqrt{2})) + 5z + z^3.$$

Second,

$$\int_{\cos\varphi + u\sin\varphi\cos\theta>0} \frac{(\cos\varphi + u\sin\varphi\cos\theta)^2}{1+u^2}\, d\theta \;=\; J(u,\cos\varphi),$$

where

$$J(u,\zeta) = \begin{cases} 0, & \text{if } \; -1 \le \zeta \le -u/\sqrt{1+u^2} \\ J_1(u,\zeta), & \text{if } \; |\zeta| < u/\sqrt{1+u^2} \\ J_2(u,\zeta), & \text{if } \; u/\sqrt{1+u^2} \le \zeta \le 1, \end{cases}$$

and

$$J_1(u,\zeta) = \frac{1}{1+u^2}\left[\theta_0\,(2\zeta^2 + u^2(1-\zeta^2)) + 3\zeta\sqrt{u^2-\zeta^2(1+u^2)}\right],$$

$$J_2(u,\zeta) = \frac{\pi}{1+u^2}\,[2\zeta^2 + u^2(1-\zeta^2)], \qquad \theta_0 = \arccos\left(-\frac{\zeta}{u\sqrt{1-\zeta^2}}\right).$$

Taking into account these formulas and changing the variable $\zeta = \cos\varphi$ in (5.8), one gets

$$p_\varepsilon(u,V) = \varepsilon\, \frac{e^{-V^2/2}}{(2\pi)^{3/2}} \int_{-1}^{1} I(\varepsilon V\zeta)\, J(u,\zeta)\, d\zeta =$$

$$= \varepsilon\, \frac{e^{-V^2/2}}{(2\pi)^{3/2}} \left(\int_{-u/\sqrt{1+u^2}}^{u/\sqrt{1+u^2}} I(\varepsilon V\zeta)\, J_1(u,\zeta)\, d\zeta + \int_{u/\sqrt{1+u^2}}^{1} I(\varepsilon V\zeta)\, J_2(u,\zeta)\, d\zeta \right).$$

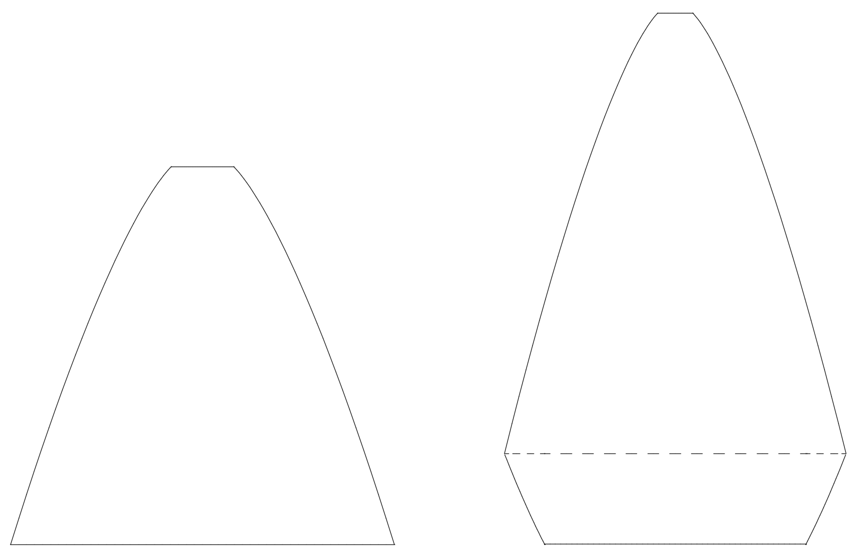
Next, one numerically calculates the function $h_*(V)$ according to the formula (4.7). This function is shown on Fig. 9(a); it divides the parameter set $\mathbb{R}^2_+$ into two subsets corresponding to two different kinds of solutions. The function $h_*(V)$ looks like linear, but is not; the graph of its derivative is shown on Fig. 9(b). The function $\tilde{\mathrm{R}}(h,V)$ is calculated according to the formulas given in subsection 4.2; the graphs of $\tilde{\mathrm{R}}(h,V)$ versus h are shown on Fig. 10 for several values of V. On Fig. 8 the examples of solutions of the first and the second kind are presented, for parameters indicated there.

Appendix A

Proof of lemma 1

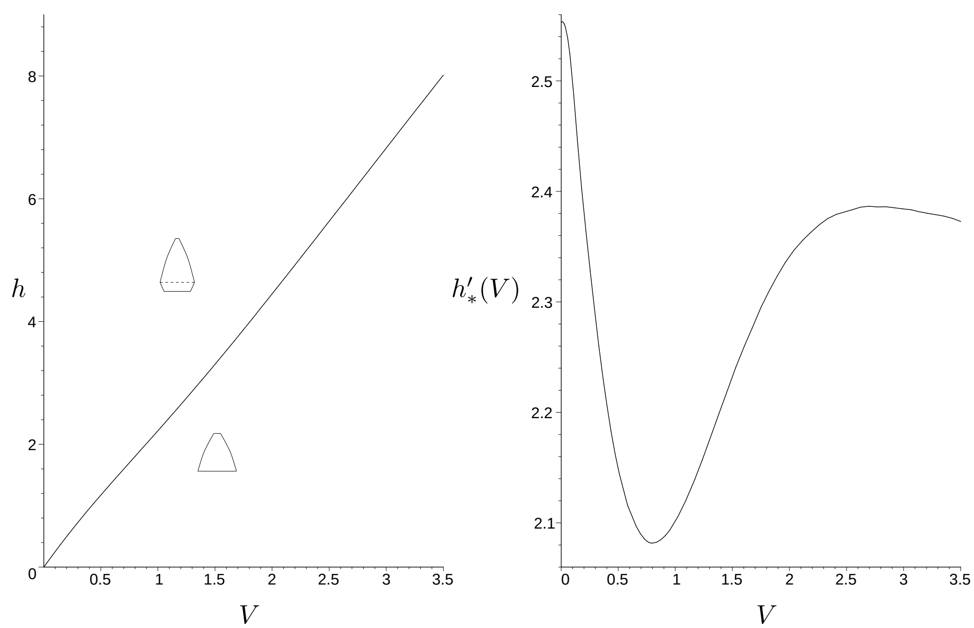
Changing the variable $v = r\nu,\ r \ge 0,\ \nu \in S^{d-1}$ in the integral (2.10) and using condition A, one obtains

$$p_\varepsilon(u) = \varepsilon \int_{S^{d-1}} d\mathcal{H}^{d-1}(\nu) \int_0^{+\infty} r^2\, \frac{(\nu_1 u + \varepsilon\nu_d)_-^2}{1+u^2}\, \rho(r\nu)\, r^{d-1} dr =$$



(a) $V = 1$, $h = 1.97$ (b) $V = 1$, $h = 3.11$

Figure 8: The solutions in three-dimensional case, for the distribution function ρ_V (5.1).



(a) $h_*(V)$

(b) dh_*/dV

Figure 9: Three-dimensional case. The function $h_*(V)$ divides the parameter plane V-h into two subsets corresponding to the two kinds of solutions.

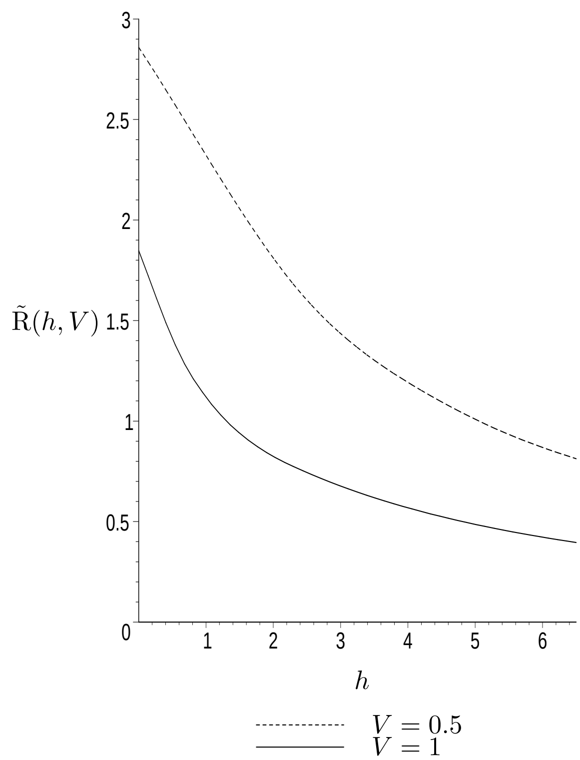


Figure 10: Three-dimensional case. Minimal reduced resistance $\tilde{\mathrm{R}}(V,h)$ versus height h of the body.

$$= \varepsilon \int_{S^{d-1}} \frac{(\nu_1 u + \varepsilon \nu_d)_-^2}{1+u^2}\ \bar\rho(\nu)\ d\mathcal{H}^{d-1}(\nu),$$

where

$$\bar\rho(\nu) := \int_0^{+\infty} r^2\, \rho(r\nu)\, r^{d-1} dr = \int_0^{+\infty} r^2\, \sigma(\sqrt{r^2 + 2rV\,\nu_d + V^2})\, r^{d-1} dr.$$

Substituting $u = \tan\varphi$, $\varphi \in [0,\, \pi/2]$, one obtains

$$p_\varepsilon(\tan\varphi) = \varepsilon \int_{S^{d-1}} (\nu_1 \sin\varphi + \varepsilon \nu_d \cos\varphi)_-^2\, \bar\rho(\nu)\, d\mathcal{H}^{d-1}(\nu).$$

Substitute "+" for ε and consider the rotation T_φ that sends the vector $(\sin\varphi,\, 0, \ldots, 0,\, \cos\varphi)$ to e_d and leaves the vectors $e_i,\ i = 2, \ldots, d-1$ unchanged. For any $\nu \in \mathbb{R}^d$ one has $T_\varphi \nu = (\cos\varphi\, \nu_1 - \sin\varphi\, \nu_d,\ \nu_2, \ldots, \nu_{d-1},\ \sin\varphi\, \nu_1 + \cos\varphi\, \nu_d)$. Changing the variable $T_\varphi \nu = \omega$, one gets

$$p_+(\tan\varphi) = \int_{S_-^{d-1}} \omega_d^2\ \bar\rho(T_\varphi^{-1}\omega)\ d\mathcal{H}^{d-1}(\omega), \tag{A.1}$$

where $S_-^{d-1} := \{\omega \in S^{d-1} : \omega_d < 0\}$. Designate

$$\varrho(z) := \int_0^\infty r^2\, \sigma(\sqrt{r^2 + 2rVz + V^2})\, r^{d-1} dr, \qquad |z| \le 1; \tag{A.2}$$

obviously, $\bar\rho(\nu) = \varrho(\nu_d)$. Using condition A, one concludes that the function ϱ is continuously differentiable, and its derivative

$$\varrho'(z) = \int_0^\infty r^2\, \frac{\sigma'(\sqrt{r^2 + 2rVz + V^2})}{\sqrt{r^2 + 2rVz + V^2}}\, rV\, r^{d-1} dr$$

is negative and monotone non-decreasing; in particular,

$$\text{as} \quad z > 0, \qquad \varrho'(z) > \varrho'(-z). \tag{A.3}$$

Using that

$$T_\varphi^{-1}\omega = T_{-\varphi}\omega = (\cos\varphi\, \omega_1 + \sin\varphi\, \omega_d,\ \omega_2, \ldots, \omega_{d-1},\ -\sin\varphi\, \omega_1 + \cos\varphi\, \omega_d),$$

from (A.1) and (A.2) one obtains

$$p_+(\tan\varphi) = \int_{S_-^{d-1}} \omega_d^2\ \varrho(-\sin\varphi\, \omega_1 + \cos\varphi\, \omega_d)\ d\mathcal{H}^{d-1}(\omega). \tag{A.4}$$

Now, substitute "$-$" for ε and consider the orthogonal reflection U_φ with respect to the hyperplane $\{\sin\frac{\varphi}{2}\ \omega_1 = \cos\frac{\varphi}{2}\ \omega_d\}$; for any $\nu \in \mathbb{R}^d$ one has

$$U_\varphi \nu = (\cos\varphi\, \nu_1 + \sin\varphi\, \nu_d,\ \nu_2, \ldots, \nu_{d-1},\ \sin\varphi\, \nu_1 - \cos\varphi\, \nu_d).$$

Changing the variable $U_\varphi \nu = \omega$, one gets

$$p_-(\tan\varphi) = -\int_{S_-^{d-1}} \omega_d^2\ \bar\rho(U_\varphi^{-1}\omega)\ d\mathcal{H}^{d-1}(\omega).$$

Using that $U_\varphi^{-1} = U_\varphi$, one obtains

$$p_-(\tan\varphi) = -\int_{S_-^{d-1}} \omega_d^2\ \varrho(\sin\varphi\,\omega_1 - \cos\varphi\,\omega_d)\ d\mathcal{H}^{d-1}(\omega). \tag{A.5}$$

The formulas (A.4) and (A.5) can be written in the unified form

$$p_\varepsilon(\tan\varphi) = \varepsilon\int_{S_-^{d-1}} \omega_d^2\ \varrho(\varepsilon(-\sin\varphi\,\omega_1 + \cos\varphi\,\omega_d))\ d\mathcal{H}^{d-1}(\omega). \tag{A.6}$$

Substituting $\varphi = \pi/2$ in (A.6), one obtains

$$\lim_{u\to+\infty} p_\varepsilon(u) = p_\varepsilon(+\infty) = \varepsilon\int_{S_-^{d-1}} \omega_d^2\,\varrho(-\varepsilon\omega_1)\,d\mathcal{H}^{d-1}(\omega).$$

Using that S_-^{d-1} is invariant with respect to reflection $\omega_1 \mapsto -\omega_1$, one concludes that $p_+(+\infty) = -p_-(+\infty)$, so (a) is proved.

Further, using (A.6), one concludes that the function p_ε is continuously differentiable, and

$$p_\varepsilon'(\tan\varphi) = \varepsilon\cos^2\varphi\cdot\int_{S_-^{d-1}} \omega_d^2\ \frac{\partial}{\partial\varphi}\ \varrho(\varepsilon(-\sin\varphi\,\omega_1 + \cos\varphi\,\omega_d))\ d\mathcal{H}^{d-1}(\omega) =$$

$$= -\cos^2\varphi\cdot\int_{S_-^{d-1}} \omega_d^2\ (\cos\varphi\,\omega_1 + \sin\varphi\,\omega_d)\ \varrho'(\varepsilon(-\sin\varphi\,\omega_1 + \cos\varphi\,\omega_d))\ d\mathcal{H}^{d-1}(\omega). \tag{A.7}$$

Substituting $\varphi = 0$ in (A.7), one obtains

$$p_\varepsilon'(0) = -\int_{S_-^{d-1}} \omega_d^2\,\omega_1\ \varrho'(\varepsilon\omega_d)\ d\mathcal{H}^{d-1}(\omega).$$

Using that S_-^{d-1} is invariant and the integrand is antisymmetric with respect to reflection $\omega_1 \mapsto -\omega_1$, one concludes that $p_\varepsilon'(0) = 0$. Next, substituting $\varphi = \pi/2$ in (A.7), one obtains

$$\lim_{u\to+\infty} p_\varepsilon'(u) = p_\varepsilon'(+\infty) = 0\cdot\int_{S_-^{d-1}} \omega_d^3\ \varrho'(-\varepsilon\omega_1)\ d\mathcal{H}^{d-1}(\omega) = 0.$$

Thus, (b) is proved.

Further, one has

$$p_+'(\tan\varphi) - p_-'(\tan\varphi) = \cos^2\varphi\cdot\int_{S_-^{d-1}} \omega_d^2\ \frac{\partial}{\partial\varphi}\ \Phi(\varphi,\omega_1,\omega_d)\ d\mathcal{H}^{d-1}(\omega), \tag{A.8}$$

$$p'_+(\tan\varphi) = \cos^2\varphi \cdot \int_{S_-^{d-1}} \omega_d^2 \, \frac{\partial}{\partial\varphi}\, \Phi_+(\varphi,\omega_1,\omega_d)\ d\mathcal{H}^{d-1}(\omega), \tag{A.9}$$

where

$$\Phi(\varphi,\omega_1,\omega_d) = \varrho(-\sin\varphi\ \omega_1 + \cos\varphi\ \omega_d) + \varrho(\sin\varphi\ \omega_1 - \cos\varphi\ \omega_d).$$

$$\Phi_+(\varphi,\omega_1,\omega_d) = \varrho(-\sin\varphi\ \omega_1 + \cos\varphi\ \omega_d).$$

Designate

$$I(c,\varphi) = \int_{\Gamma_c} \omega_d^2 \, \frac{\partial}{\partial\varphi}\, \Phi(\varphi,\omega_1,\omega_d)\ d\mathcal{H}^1(\omega_1,\omega_d),$$

$$I_+(c,\varphi) = \int_{\Gamma_c} \omega_d^2 \, \frac{\partial}{\partial\varphi}\, \Phi_+(\varphi,\omega_1,\omega_d)\ d\mathcal{H}^1(\omega_1,\omega_d),$$

where $\Gamma_c = \{(\omega_1,\omega_d) : \omega_1^2 + \omega_d^2 = c^2,\ \omega_d < 0\}$. Let us prove that

$$\text{for any } c \in (0,\,1) \text{ and } \varphi \in (0,\,\pi/2), \quad I(c,\varphi) < 0 \ \text{ and } \ I_+(c,\varphi) < 0; \tag{A.10}$$

then, integrating $I(1-|\tilde\omega|^2,\varphi)$ and $I_+(1-|\tilde\omega|^2,\varphi)$ over $\tilde\omega = (\omega_2,\dots,\omega_{d-1})$ and multiplying by $\cos^2\varphi$, one will conclude that the right hand sides of (A.8) and of (A.9) are negative, and so, (c) and the first inequality in (d) are true.

Parametrize the curve Γ_c by $\omega_1 = c\cos\theta,\ \omega_d = -c\sin\theta,\ \theta \in [0,\,\pi]$, then

$$I(c,\varphi) = \int_0^\pi c^2\sin^2\theta\, \frac{\partial}{\partial\varphi} \big[\varrho(c\sin(\varphi+\theta)) + \varrho(-c\sin(\varphi+\theta))\big]\, c\, d\theta = c^3\mathcal{I}_1 + c^3\mathcal{I}_2,$$

where

$$\mathcal{I}_1 = \int_0^{\pi-2\varphi} \sin^2\theta\, \frac{\partial}{\partial\varphi} \big[\dots\big]\, d\theta, \tag{A.11}$$

$$\mathcal{I}_2 = \int_{\pi-2\varphi}^{\pi} \sin^2\theta\, \frac{\partial}{\partial\varphi} \big[\dots\big]\, d\theta, \tag{A.12}$$

and

$$I_+(c,\varphi) = \int_0^\pi c^2\sin^2\theta\, \frac{\partial}{\partial\varphi}\, \varrho(-c\sin(\varphi+\theta))\, c\, d\theta = c^3\mathcal{I}_1^+ + c^3\mathcal{I}_2^+,$$

where

$$\mathcal{I}_1^+ = \int_0^{\pi-2\varphi} \sin^2\theta\, \frac{\partial}{\partial\varphi}\, \varrho(-c\sin(\varphi+\theta))\, d\theta, \tag{A.13}$$

$$\mathcal{I}_2^+ = \int_{\pi-2\varphi}^{\pi} \sin^2\theta\, \frac{\partial}{\partial\varphi}\, \varrho(-c\sin(\varphi+\theta))\, d\theta. \tag{A.14}$$

Changing the variable $\psi = \theta + \varphi - \pi/2$ in (A.11), one obtains

$$\mathcal{I}_1 = \int_{-\pi/2+\varphi}^{\pi/2-\varphi} \cos^2(\varphi-\psi)\, \frac{d}{d\psi} \big[\varrho(c\cos\psi) + \varrho(-c\cos\psi)\big]\, d\psi,$$

and using the fact that the function $\frac{d}{d\psi}[\ldots]$ under the sign of integral is odd, one gets

$$\mathcal{I}_1 = \int_0^{\pi/2-\varphi} (\cos^2(\varphi-\psi) - \cos^2(\varphi+\psi)) \, \frac{d}{d\psi} \left[\varrho(c\cos\psi) + \varrho(-c\cos\psi) \right] d\psi.$$

One has $\cos^2(\varphi-\psi) - \cos^2(\varphi+\psi) = \sin 2\varphi \sin 2\psi > 0$. Taking into account (A.3), one also has that $\frac{d}{d\psi}\left[\ldots\right] = -c\sin\psi \; (\varrho'(c\cos\psi) - \varrho'(-c\cos\psi)) < 0$. Hence, $\mathcal{I}_1 < 0$.

Making the same change of variable in (A.13), one gets

$$\mathcal{I}_1^+ = \int_{-\pi/2+\varphi}^{\pi/2-\varphi} \cos^2(\varphi-\psi) \, \frac{d}{d\psi} \, \varrho(-c\cos\psi) \, d\psi =$$

$$= \int_0^{\pi/2-\varphi} (\cos^2(\varphi-\psi) - \cos^2(\varphi+\psi)) \, \frac{d}{d\psi} \, \varrho(-c\cos\psi) \, d\psi.$$

One has $\cos^2(\varphi-\psi) - \cos^2(\varphi+\psi) > 0$, and $\frac{d}{d\psi} \, \varrho(-c\cos\psi) = c\sin\psi \, \varrho'(-c\cos\psi)) < 0$, thus $\mathcal{I}_1^+ < 0$.

On the other hand, changing the variable $\chi = \theta + \varphi - \pi$ in (A.12), one obtains

$$\mathcal{I}_2 = \int_{-\varphi}^{\varphi} \sin^2(\varphi-\chi) \, \frac{d}{d\chi} \left[\varrho(c\sin\chi) + \varrho(-c\sin\chi) \right] d\chi.$$

The function $\frac{d}{d\chi}[\ldots]$ is odd, therefore

$$\mathcal{I}_2 = \int_0^{\varphi} (\sin^2(\varphi-\chi) - \sin^2(\varphi+\chi)) \, \frac{d}{d\chi} \left[\varrho(c\sin\chi) + \varrho(-c\sin\chi) \right] d\chi.$$

One has $\sin^2(\varphi-\chi) - \sin^2(\varphi+\chi) = -\sin 2\varphi \sin 2\chi < 0$, and $\frac{d}{d\chi}\left[\ldots\right] = c\cos\chi \, (\varrho'(c\sin\chi) - \varrho'(-c\sin\chi)) > 0$. Hence, $\mathcal{I}_2 < 0$.

Further, as $\pi - 2\varphi \le \theta \le \pi$, one has $\frac{\partial}{\partial\varphi} \, \varrho(-c\sin(\varphi+\theta)) = -c\cos(\varphi+\theta) \, \varrho'(-c\sin(\varphi+\theta)) < 0$, and using (A.14), one concludes that $\mathcal{I}_2^+ < 0$.

Thus, the inequalities in (A.10) are proved, and so, (c) and the first inequality in (d) are true.

Passing to the limit $\varphi \to \pi/2$ in (A.5), one gets

$$\lim_{\varphi\to\pi/2} p_-(\tan\varphi) = p_-(+\infty) = -\int_{S_-^{d-1}} \omega_d^2 \; \varrho(\omega_1) \; d\mathcal{H}^{d-1}(\omega).$$

Thus, to prove the second inequality in (d), one needs to verify that

$$\int_{S_-^{d-1}} \omega_d^2 \; \varrho(\omega_1) \; d\mathcal{H}^{d-1}(\omega) > \int_{S_-^{d-1}} \omega_d^2 \; \varrho(\sin\varphi\,\omega_1 - \cos\varphi\,\omega_d) \; d\mathcal{H}^{d-1}(\omega).$$

Denote

$$J(c,\varphi) = \int_{\Gamma_c} \omega_d^2 \, \varrho(\sin\varphi\,\omega_1 - \cos\varphi\,\omega_d) \, d\mathcal{H}^1(\omega_1,\omega_d) = c^3 \int_0^{\pi} \sin^2\theta \, \varrho(c\sin(\varphi+\theta)) \, d\theta.$$

It suffices to prove that

$$\text{for any } c \in (0,\,1) \text{ and } \varphi \in (0,\,\pi/2), \qquad J(c,\pi/2) > J(c,\varphi); \tag{A.15}$$

then by integrating $J(1-|\tilde\omega|^2,\pi/2)$ and $J(1-|\tilde\omega|^2,\varphi)$ over $\tilde\omega = (\omega_2,\ldots,\omega_{d-1})$, the inequality (A.15) will be established.

One has $J(c,\varphi) = J_1 + J_2$, $J(c,\pi/2) = J_1^* + J_2^*$, where

$$J_1 = \int_0^{\pi/2-\varphi} \sin^2\theta\, \varrho(c\sin(\varphi+\theta))\, d\theta, \qquad J_2 = \int_{\pi/2-\varphi}^{\pi} \sin^2\theta\, \varrho(c\sin(\varphi+\theta))\, d\theta,$$

$$J_1^* = \int_0^{\pi/2+\varphi} \sin^2\theta\, \varrho(c\cos\theta)\, d\theta, \qquad J_2^* = \int_{\pi/2+\varphi}^{\pi} \sin^2\theta\, \varrho(c\cos\theta)\, d\theta.$$

As $0 < \theta < \pi/2-\varphi$, one has $-\cos\theta < 0 < \sin(\varphi+\theta)$, hence $\varrho(-c\cos\theta) > \varrho(c\sin(\varphi+\theta)$, thus

$$J_2^* = \int_0^{\pi/2-\varphi} \sin^2\theta\, \varrho(-c\cos\theta)\, d\theta > \int_0^{\pi/2-\varphi} \sin^2\theta\, \varrho(c\sin(\varphi+\theta))\, d\theta = J_1. \tag{A.16}$$

Further, one has

$$2J_1^* = \int_0^{\pi/2+\varphi} \sin^2\theta\, \varrho(c\cos\theta)\, d\theta + \int_0^{\pi/2+\varphi} \sin^2(\pi/2+\varphi-\theta)\, \varrho(c\cos(\pi/2+\varphi-\theta)\, d\theta,$$

$$2J_2 = \int_0^{\pi/2+\varphi} \sin^2\theta\, \varrho(c\sin(\theta-\varphi))\, d\theta + \int_0^{\pi/2+\varphi} \sin^2(\pi/2+\varphi-\theta)\, \varrho(c\sin(\pi/2-\theta))\, d\theta,$$

hence

$$2J_1^* - 2J_2 = \int_0^{\pi/2+\varphi} [\sin^2\theta - \cos^2(\theta-\varphi)][\varrho(c\cos\theta) - \varrho(c\sin(\theta-\varphi)]\, d\theta. \tag{A.17}$$

Taking into account that the function ϱ is monotone decreasing and that $\sin^2\theta - \cos^2(\theta-\varphi) = (\sin(\theta-\varphi)-\cos\theta)(\sin(\theta-\varphi)+\cos\theta)$, one concludes that the integrand in (A.17) is positive, hence $J_1^* > J_2$. From here and from (A.16) it follows that (A.15) is true. Lemma 1 is completely proved. $\square$

Proof of lemma 2

(a) Parametrize the set S^1_- according to $\nu_1 = -\sin\theta$, $\nu_2 = -\cos\theta$, $\theta \in [-\pi/2,\,\pi/2]$, then (A.6) takes the form

$$p_\varepsilon(\tan\varphi) = \varepsilon \int_{-\pi/2}^{\pi/2} \cos^2\theta\ \varrho(-\varepsilon\cos(\varphi+\theta))\ d\theta, \qquad \varepsilon \in \{-,\ +\}. \tag{A.18}$$

Twice differentiating both parts of this equation with respect to φ, one obtains

$$\frac{p''_\varepsilon(\tan\varphi)}{\cos^3\varphi} = -\varepsilon \int_{-\pi/2}^{\pi/2} \cos^2\theta\ 2\sin\varphi\ \frac{\partial}{\partial\varphi}\, \varrho(-\varepsilon\cos(\varphi+\theta))\ d\theta +$$

$$+\varepsilon \int_{-\pi/2}^{\pi/2} \cos^2\theta \ \cos\varphi \ \frac{\partial^2}{\partial\varphi^2}\,\varrho(-\varepsilon\cos(\varphi+\theta))\ d\theta.$$

Integrating the second integral by parts and taking into account that $\frac{\partial^k}{\partial\varphi^k}\,\varrho(-\varepsilon\cos(\varphi+\theta)) = \frac{\partial^k}{\partial\theta^k}\,\varrho(-\varepsilon\cos(\varphi+\theta)),\ k=1,\ 2$, one gets

$$\frac{p''_\varepsilon(\tan\varphi)}{\cos^3\varphi} = \varepsilon\int_{-\pi/2}^{\pi/2} 2\cos\theta\ \sin(\theta-\varphi)\ \frac{\partial}{\partial\theta}\,\varrho(-\varepsilon\cos(\varphi+\theta))\,d\theta.$$

Integrating by parts once more and denoting $g_\varepsilon(\varphi) := p''_\varepsilon(\tan\varphi)/(2\cos^3\varphi)$, one gets

$$g_\varepsilon(\varphi) = -\varepsilon\int_{-\pi/2}^{\pi/2} \varrho(-\varepsilon\cos(\varphi+\theta))\ \cos(2\theta-\varphi)\,d\theta. \tag{A.19}$$

Fix the sign "+" and prove that
(I) for $0<\varphi<\pi/6,\ g_+(\varphi)<0$;
(II) for $\varphi\ge 0.3\pi,\ g_+(\varphi)>0$;
(III) for $\pi/6\le\varphi\le 0.3\pi,\ g'_+(\varphi)>0$.

The relations (I), (II), and (III) imply that there exists $\bar u_+\in(1/\sqrt{3},\ \tan(0.3\pi))$ such that $p''_+(u)<0$ as $u\in(0,\ \bar u_+)$, and $p''_+(u)>0$ as $u\in(\bar u_+,\ +\infty)$.

(I) Changing the variable $\psi=\theta-\varphi/2+\pi/4$, one gets

$$g_+(\varphi) = -\int_{-\pi/4-\varphi/2}^{3\pi/4-\varphi/2} \varrho(-\cos(3\varphi/2-\pi/4+\psi))\ \sin 2\psi\ d\psi \tag{A.20}$$

$$= \mathcal{L}_1+\mathcal{L}_2, \qquad \text{where} \quad \mathcal{L}_1 = -\int_{-\pi/4-\varphi/2}^{\pi/4+\varphi/2}\ldots, \quad \mathcal{L}_2 = -\int_{\pi/4+\varphi/2}^{3\pi/4-\varphi/2}\ldots.$$

One has

$$\mathcal{L}_1 = \int_0^{\pi/4+\varphi/2}\left[\varrho\left(-\cos\left(3\varphi/2-\pi/4-\psi\right)\right)-\varrho\left(-\cos\left(3\varphi/2-\pi/4+\psi\right)\right)\right]\,\sin 2\psi\,d\psi. \tag{A.21}$$

One has $0\le 2\psi\le\pi/2+\varphi\le\pi$, hence $\sin 2\psi\ge 0$. Using that $0<\varphi<\pi/6$, one obtains that $-\pi/2\le 3\varphi/2-\pi/4-\psi < -|3\varphi/2-\pi/4+\psi|$, therefore $-\cos(3\varphi/2-\pi/4-\psi) > -\cos(3\varphi/2-\pi/4+\psi)$. The function ϱ monotone decreases, therefore $\varrho\left(-\cos(3\varphi/2-\pi/4-\psi)\right) < \varrho\left(-\cos(3\varphi/2-\pi/4+\psi)\right)$. Thus, the integrand in (A.21) is negative, and so, $\mathcal{L}_1<0$.

Change the variable $\chi=\psi-\pi/2$ in the integral $\mathcal{L}_2$. One obtains

$$\mathcal{L}_2 = \int_{-\pi/4+\varphi/2}^{\pi/4-\varphi/2}\varrho\left(-\cos\left(3\varphi/2+\pi/4+\chi\right)\right)\,\sin 2\chi\,d\chi =$$

$$= \int_0^{\pi/4-\varphi/2}\left[\varrho\left(-\cos\left(3\varphi/2+\pi/4+\chi\right)\right)-\varrho\left(-\cos\left(3\varphi/2+\pi/4-\chi\right)\right)\right]\,\sin 2\chi\,d\chi. \tag{A.22}$$

One has $0 \le 3\varphi/2+\pi/4-\chi \le 3\varphi/2+\pi/4+\chi \le \pi$, hence $-\cos(3\varphi/2+\pi/4-\chi) \le -\cos(3\varphi/2+\pi/4+\chi)$, and so, $\varrho(-\cos(3\varphi/2+\pi/4-\chi)) \ge \varrho(-\cos(3\varphi/2+\pi/4+\chi))$. Therefore the integrand in (A.22) is negative, and $\mathcal{L}_2 \le 0$. Thus, (I) is proved.

(II) By (A.20), one has

$$g_+(\varphi) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4,$$

where

$$\mathcal{I}_1 = -\int_{-\varphi}^{\varphi}(...),\ \ \mathcal{I}_2 = -\int_{-\pi/4-\varphi/2}^{-\varphi}(...),\ \ \mathcal{I}_3 = -\int_{\varphi}^{\pi/4+\varphi/2}(...),\ \ \mathcal{I}_4 = -\int_{\pi/4+\varphi/2}^{3\pi/4-\varphi/2}(...),$$

and $(...) = \varrho(-\cos(3\varphi/2-\pi/4+\psi))\,\sin 2\psi\, d\psi$.

One has

$$\mathcal{I}_1 = \int_0^{\varphi}[\varrho(-\cos(3\varphi/2-\pi/4-\psi))-\varrho(-\cos(3\varphi/2-\pi/4+\psi))]\,\sin 2\psi\, d\psi; \quad (A.23)$$

using that $\varphi \ge 0.3\pi$, one easily verifies that as $0<\psi<\varphi$, $|3\varphi/2-\pi/4-\psi| < 3\varphi/2-\pi/4+\psi \le \pi$, hence $-\cos(3\varphi/2-\pi/4-\psi) < -\cos(3\varphi/2-\pi/4+\psi)$. Using that ϱ monotone decreases, one concludes that the integrand in (A.23) is positive, thus $\mathcal{I}_1 > 0$.

Next, as $-\pi/4-\varphi/2 \le \psi \le -\varphi$, one has $\sin 2\psi \le 0$ and $|3\varphi/2-\pi/4+\psi| \le 3\varphi/2-\pi/4+\psi+2\varphi \le \pi$, hence $-\cos(3\varphi/2-\pi/4+\psi) \le -\cos(3\varphi/2-\pi/4+\psi+2\varphi)$, and thus, $\varrho(-\cos(3\varphi/2-\pi/4+\psi)) \le \varrho(-\cos(3\varphi/2-\pi/4+\psi+2\varphi))$. Therefore,

$$\mathcal{I}_2 \ge -\int_{-\pi/4-\varphi/2}^{-\varphi}\varrho(-\cos(3\varphi/2-\pi/4+\psi+2\varphi))\,\sin 2\psi\, d\psi =$$

$$= \int_{\varphi}^{\pi/4+\varphi/2}\varrho(-\cos(3\varphi/2-\pi/4-\chi+2\varphi))\,\sin 2\chi\, d\chi\,,$$

and

$$\mathcal{I}_2+\mathcal{I}_3 \ge \int_{\varphi}^{\pi/4+\varphi/2}[\varrho\,(-\cos(3\varphi/2-\pi/4-\psi+2\varphi))-$$

$$-\varrho\,(-\cos(3\varphi/2-\pi/4+\psi))]\,\sin 2\psi\, d\psi. \quad (A.24)$$

On the other hand, one has

$$\mathcal{I}_4 = \int_{\pi/2}^{3\pi/4-\varphi/2}[\varrho\,(-\cos(3\varphi/2+3\pi/4-\psi))-\varrho\,(-\cos(3\varphi/2-\pi/4+\psi))]\,\sin 2\psi\, d\psi. \quad (A.25)$$

Changing the variable $\theta = \psi-\varphi$ in (A.24) and $\theta = \psi-\pi/2$ in (A.25) and summing both parts of these relations, one obtains

$$\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4 \ge \int_0^{\pi/4-\varphi/2}\Psi(\theta)\, d\theta,$$

where

$$\Psi(\theta) = [\varrho\,(-\cos(5\varphi/2 - \pi/4 - \theta)) - \varrho\,(-\cos(5\varphi/2 - \pi/4 + \theta))]\,\sin(2\theta + 2\varphi) -$$

$$-[\varrho\,(-\cos(3\varphi/2 + \pi/4 - \theta)) - \varrho\,(-\cos(3\varphi/2 + \pi/4 + \theta))]\,\sin 2\theta.$$

Let us show that $\Psi(\theta) \geq 0$; it will follow that $\mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 \geq 0$, and thus, (II) will be proved.

One has $0 \leq 2\theta \leq \pi/2 - \varphi,\ \pi/2 - \varphi \leq 2\varphi \leq 2\theta + 2\varphi \leq \pi/2 + \varphi$, hence

$$0 \leq \sin 2\theta \leq \sin(2\theta + 2\varphi). \tag{A.26}$$

Denote $J_1(\theta) = \varrho\,(-\cos(5\varphi/2 - \pi/4 - \theta)) - \varrho\,(-\cos(5\varphi/2 - \pi/4 + \theta)),\ J_2(\theta) = \varrho\,(-\cos(3\varphi/2 + \pi/4 - \theta)) - \varrho\,(-\cos(3\varphi/2 + \pi/4 + \theta))$. One has

$$J_1(\theta) = -\int_{-\theta}^{\theta} \varrho'(-\cos(5\varphi/2 - \pi/4 + \chi))\,\sin(5\varphi/2 - \pi/4 + \chi)\,d\chi\,,$$

$$J_2(\theta) = -\int_{-\theta}^{\theta} \varrho'(-\cos(3\varphi/2 + \pi/4 + \chi))\,\sin(3\varphi/2 + \pi/4 + \chi)\,d\chi\,.$$

Using that $\varphi \geq 0.3\pi$, one gets $\pi - 2\varphi \leq 3\varphi - \pi/2 \leq 5\varphi/2 - \pi/4 + \chi \leq 2\varphi$ and $2\varphi \leq 3\varphi/2 + \pi/4 + \chi \leq \varphi + \pi/2 \leq \pi$, hence

$$\sin(5\varphi/2 - \pi/4 + \chi) \geq \sin(3\varphi/2 + \pi/4 + \chi) \geq 0 \tag{A.27}$$

and $-\cos(5\varphi/2 - \pi/4 + \chi) \leq -\cos(3\varphi/2 + \pi/4 + \chi)$. Taking into account that ϱ' is negative and monotone increasing, one gets

$$\varrho'(-\cos(5\varphi/2 - \pi/4 + \chi)) \leq \varrho'(-\cos(3\varphi/2 + \pi/4 + \chi)) \leq 0. \tag{A.28}$$

From (A.27) and (A.28) it follows that $J_1(\theta) \geq J_2(\theta) \geq 0$, and taking into account (A.26), one concludes that $\Psi(\theta) \geq 0$.

(III) One has

$$g_+'(\varphi) = -\frac{d}{d\varphi}\int_{-\pi/4-\varphi/2}^{3\pi/4-\varphi/2} \varrho(-\cos(3\varphi/2 - \pi/4 + \psi))\,\sin 2\psi\,d\psi =$$

$$= -\int_{-\pi/4-\varphi/2}^{3\pi/4-\varphi/2} \frac{\partial}{\partial\varphi}\,\varrho(-\cos(3\varphi/2 - \pi/4 + \psi))\,\sin 2\psi\,d\psi +$$

$$+\frac{1}{2}\cos\varphi\,[\varrho(-\cos(\varphi - \pi/2)) - \varrho(-\cos(\varphi + \pi/2))].$$

One has $\cos\varphi > 0,\ -\cos(\varphi - \pi/2) < 0 < -\cos(\varphi + \pi/2)$, hence $\varrho(-\cos(\varphi - \pi/2)) - \varrho(-\cos(\varphi + \pi/2)) > 0$, and thus,

$$g_+'(\varphi) > -\int_{-\pi/4-\varphi/2}^{3\pi/4-\varphi/2} \frac{\partial}{\partial\varphi}\,\varrho(-\cos(3\varphi/2 - \pi/4 + \psi))\,\sin 2\psi\,d\psi = -(\mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 + \mathcal{K}_4),$$

where

$$\mathcal{K}_1 = \int_{-\pi/4-\varphi/2}^{\pi/2-3\varphi} ..., \;\; \mathcal{K}_2 = \int_{\pi/2-3\varphi}^{0} ..., \;\; \mathcal{K}_3 = \int_{0}^{\pi/4+\varphi/2} ..., \;\; \mathcal{K}_4 = \int_{\pi/4+\varphi/2}^{3\pi/4-\varphi/2} ... .$$

In order to prove (III), it suffices to show that $\mathcal{K}_1 \le 0,\ \mathcal{K}_2 \le 0,\ \mathcal{K}_3 \le 0,$ and $\mathcal{K}_4 \le 0$.

One has

$$\mathcal{K}_1 = \frac{3}{2}\int_{-\pi/4-\varphi/2}^{\pi/2-3\varphi} \varrho'(-\cos(3\varphi/2-\pi/4+\psi))\,\sin(3\varphi/2-\pi/4+\psi)\,\sin 2\psi\, d\psi.$$

Using that $\varphi \le 0.3\pi$, one easily verifies that $-\pi/4-\varphi/2 \le \pi/2-3\varphi$. For $-\pi/4-\varphi/2 \le \psi \le \pi/2-3\varphi$, one has $-\pi/2 \le 3\varphi/2-\pi/4+\psi \le 0,\ -\pi \le 2\psi \le 0$, hence $\sin(3\varphi/2-\pi/4+\psi) \le 0,\ \sin 2\psi \le 0$, and taking into account that $\varrho' \le 0$, one concludes that $\mathcal{K}_1 \le 0$.

Changing the variable $\psi = 3\varphi/2-\pi/4+\psi$, one has

$$\mathcal{K}_2 = \frac{3}{2}\int_{\pi/4-3\varphi/2}^{3\varphi/2-\pi/4} \frac{d}{d\chi}\,\varrho(-\cos\chi)\,\cos(2\chi-3\varphi)\,d\chi =$$

$$= \frac{3}{2}\int_{0}^{3\varphi/2-\pi/4} \frac{d}{d\chi}\,\varrho(-\cos\chi)\,[\cos(2\chi-3\varphi)-\cos(2\chi+3\varphi)]\,d\chi.$$

One has $\frac{d}{d\chi}\,\varrho(-\cos\chi) \le 0,\ \cos(2\chi-3\varphi)-\cos(2\chi+3\varphi) = 2\sin 2\chi \sin 3\varphi \ge 0$, hence $\mathcal{K}_2 \le 0$.

Further, one has

$$\mathcal{K}_3 = \frac{3}{2}\int_{0}^{\pi/4+\varphi/2} \varrho'(-\cos(3\varphi/2-\pi/4+\psi))\,\sin(3\varphi/2-\pi/4+\psi)\,\sin 2\psi\, d\psi.$$

It is easy to verify that the integrand is negative, hence $\mathcal{K}_3 \le 0$.

Changing the variable $\theta = \psi - \pi/2$ in the integral $\mathcal{K}_4$, one obtains

$$\mathcal{K}_4 = -\int_{-\pi/4+\varphi/2}^{\pi/4-\varphi/2} \frac{\partial}{\partial\varphi}\,\varrho(-\cos(3\varphi/2+\pi/4+\theta))\,\sin 2\theta\, d\theta =$$

$$= \frac{3}{2}\int_{0}^{\pi/4-\varphi/2} [\varrho'(-\cos(3\varphi/2+\pi/4-\theta))\,\sin(3\varphi/2+\pi/4-\theta)-$$

$$-\varrho'(-\cos(3\varphi/2+\pi/4+\theta))\,\sin(3\varphi/2+\pi/4+\theta)]\,\sin 2\theta\, d\theta.$$

Using that $\varphi \ge \pi/6$, one easily verifies that $(3\varphi/2+\pi/4+\theta)-\pi/2 \ge |(3\varphi/2+\pi/4-\theta)-\pi/2|$, hence $\sin(3\varphi/2+\pi/4+\theta) \le \sin(3\varphi/2+\pi/4-\theta)$ and $-\cos(3\varphi/2+\pi/4+\theta) \ge -\cos(3\varphi/2+\pi/4-\theta)$, therefore $0 \le -\varrho'(-\cos(3\varphi/2+\pi/4+\theta)) \le -\varrho'(-\cos(3\varphi/2+\pi/4-\theta))$. This implies that $\mathcal{K}_4 \le 0$.

(b) Let us slightly change the notation just introduced: we shall write down

$$\varrho(z,V) = \int_0^\infty r^3\,\sigma(\sqrt{r^2+2rVz+V^2})\,dr,$$

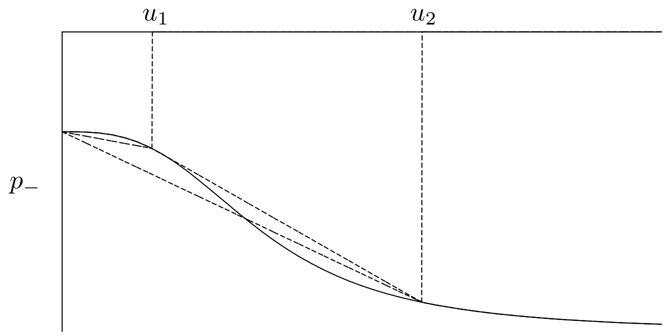


Figure 11:

$$p_-(\tan\varphi, V) = -\int_{-\pi/2}^{\pi/2} \cos^2\theta\ \varrho(\cos(\varphi+\theta), V)\ d\theta,$$

$$g_-(\varphi, V) = \frac{d^2}{du^2}\rfloor_{u=\tan\varphi} p_-(u, V)/(2\cos^3\varphi) = \int_{-\pi/2}^{\pi/2} \varrho(\cos(\varphi+\theta), V)\ \cos(2\theta-\varphi)\, d\theta,$$

thus explicitly indicating dependence of these functions on V. Using these formulas, we see that the function $\sigma^{\alpha,\beta}(r) = \sigma(r) + \alpha\,\sigma(\beta r)$ generates the function $\varrho^{\alpha,\beta}(z) = \varrho(z, V) + \frac{\alpha}{\beta^4}\varrho(z, \beta V)$ and the pressure function $p_-^{\alpha,\beta}(u) = p_-(u, V) + \frac{\alpha}{\beta^4} p_-(u, \beta V)$.

Suppose that the set $\mathcal{O}^0 := \{u : p_-(u, V) > \bar{p}_-(u, V)\}$ coincides with an interval $(0,\, u_-^0)$; otherwise, the hypothesis of lemma 2 (b) is valid for $\alpha = 0$ and arbitrary $\beta > 0$. Consider two arbitrary values $u_1 \in (0,\, u_-^0)$ and $u_2 > u_-^0$, and designate

$$\Delta_1(p) = \frac{p_-(u_1, V) - p_-(0, V)}{u_1}\,, \qquad \Delta_2(p) = \frac{p_-(u_2, V) - p_-(0, V)}{u_2}\,,$$

$$\Delta_{12}(p) = \frac{p_-(u_2, V) - p_-(u_1, V)}{u_2 - u_1}\,;$$

one has $\Delta_1(p) > \Delta_2(p) > \Delta_{12}(p)$ and $p_-'(u_2, V) > \Delta_2(p)$ (see Fig. 11).

Note that the function $g_-(\varphi, V)$ is continuous with respect to φ on $[0,\, \pi/2)$, and has a limit as $\varphi \to \pi/2 - 0$, hence the value $\mathrm{g} := \sup_{\varphi\in[0,\pi/2)} |g_-(\varphi, V)|$ is finite. Denote

$$\omega = \min\left\{ \frac{\Delta_1 p - \Delta_{12} p}{2},\ \frac{p_-'(u_2, V) - \Delta_2 p}{2} \right\}.$$

It is easy to see that if the functions $\check{p}(u) := \frac{\alpha}{\beta^4} p_-(u, \beta V)$ and $\check{g}(\varphi) := \frac{\alpha}{\beta^4}\, \frac{1}{\cos^3\varphi}\, \frac{d^2}{du^2}\rfloor_{u=\tan\varphi} p_-(u, \beta V)$ satisfy the inequalities

$$|\check{p}'(u)| < \omega \quad \text{for} \quad 0 \le u \le u_2, \qquad \check{p}'(u) > -\omega \quad \text{for} \quad u > u_2, \tag{A.29}$$

and

$$\check{g}(\varphi) < -\mathrm{g} \quad \text{for some} \quad \varphi > \arctan u_2, \tag{A.30}$$

then the function $p_-^{\alpha,\beta}(u) = p_-(u) + \check{p}(u)$ has the following property: the set $\mathcal{O}^{\alpha,\beta} = \{u : p_-^{\alpha,\beta}(u) > \bar{p}_-^{\alpha,\beta}(u)\}$ has at least two connected components; one of them is contained in $(0,\, u_2)$, and the second one contains $\tan\varphi$ and is contained in $(u_2,\, +\infty)$.

Denote

$$P^\beta(\varphi) = \max\{\sup_{0\le u\le \tan\varphi} |p_-'(u,\beta V)|,\ \sup_{u>\tan\varphi} (-p_-'(u,\beta V))\}. \tag{A.31}$$

The rest of the text (till the end of Appendix A) is devoted to the proof of the fact that for any $\varepsilon > 0$ there exist $\phi,\ \varphi$, and β such that

$$\pi/2 - \varepsilon \le \phi \le \pi/2 \quad \text{and} \quad 0 < \frac{P^\beta(\varphi)}{-g_-(\varphi,\beta V)} < \varepsilon; \tag{A.32}$$

then, letting $\phi = \tan u_2,\ \varepsilon = \omega/\mathrm{g}$, one can find α such that (A.29) and (A.30) are valid, so the statement of lemma 2 (b) for given α and β is fulfilled.

Let us carry out some auxiliary calculation. For $\varphi \in (\pi/10,\, \pi/2),\ \beta > 0$ define $R(\varphi,\beta)$ by

$$\frac{R}{\beta V} = \sin\left(\frac{5\pi}{8} - \frac{5\varphi}{4}\right). \tag{A.33}$$

Then pick out constants $\alpha > 1,\ c > 0$ in such a way that there exists a circular sector Σ of angle c, outer radius R, and inner radius R/α, with the center $(0, -\beta V)$, that belongs to the angle $\mathcal{U} := \{v = (v_1, v_2) :\ v_1 > 0,\ -\tan(\frac{5\pi}{8} - \frac{5\varphi}{4}) \le \frac{v_1}{v_2} \le -\tan(\frac{\pi}{2} - \varphi)\}$ (see Fig. 12). One can choose, in particular, arbitrary $c \in (0,\ \pi/5)$ and $\alpha = \frac{\cos(c/2)}{\cos(\pi/10)}$.

Next, using the definition of ϱ and passing to the variables $v_1 = r\sin\chi$, $v_2 = r\cos\chi$, one obtains

$$\int_{\frac{5\varphi}{4}+\frac{3\pi}{8}}^{\varphi+\frac{\pi}{2}} \varrho(\cos\chi, \beta V)\, d\chi = \iint_{\mathcal{U}} (v_1^2 + v_2^2)\, \sigma(|v + \beta V e_2|)\, dv_1 dv_2. \tag{A.34}$$

Taking into account that for $v \in \Sigma$ one has $v_1^2 + v_2^2 \ge (\beta V - R)^2$, $\beta V = R/\sin(\frac{5\pi}{8} - \frac{5\varphi}{4})$, and $\Sigma \subset \mathcal{U}$, and passing to polar coordinates, one gets

$$\iint_{\mathcal{U}} (v_1^2 + v_2^2)\, \sigma(|v + \beta V e_2|)\, dv_1 dv_2 \ge$$

$$\ge R^2 \left(\frac{1}{\sin^2(\frac{5\pi}{8} - \frac{5\varphi}{4})} - 1 \right) \iint_{\Sigma} \sigma(|v + \beta V e_2|)\, dv_1 dv_2 =$$

$$= R^2 \left(\frac{1}{\sin^2(\frac{5\pi}{8} - \frac{5\varphi}{4})} - 1 \right) \cdot c \int_{R/\alpha}^{R} \sigma(r)\, r\, dr \ge$$

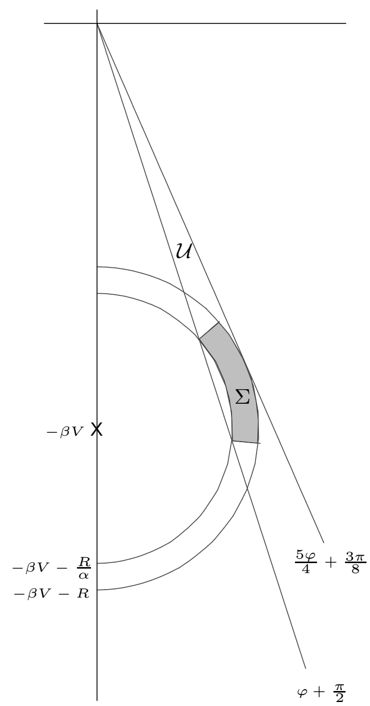

$\mathcal{U}$
Σ
$-\beta V$
$-\beta V - \frac{R}{\alpha}$
$-\beta V - R$
$\frac{5\varphi}{4} + \frac{3\pi}{8}$
$\varphi + \frac{\pi}{2}$

Figure 12:

$$\geq \left(\frac{1}{\sin^2(\frac{5\pi}{8} - \frac{5\varphi}{4})} - 1 \right) \cdot c \int_{R/\alpha}^{R} r^3 \, \sigma(r) \, dr. \tag{A.35}$$

From (A.34) and (A.35) one concludes that

$$\int_{\frac{5\varphi}{4} + \frac{3\pi}{8}}^{\varphi + \frac{\pi}{2}} \varrho(\cos\chi, \beta V) \, d\chi \geq \left(\frac{1}{\sin^2(\frac{5\pi}{8} - \frac{5\varphi}{4})} - 1 \right) \cdot c \int_{R/\alpha}^{R} r^3 \, \sigma(r) \, dr. \tag{A.36}$$

Besides, one has

$$\int_{\varphi - \frac{\pi}{2}}^{\frac{3\varphi}{2} + \frac{\pi}{4}} \varrho(\cos\chi, \beta V) \, d\chi \leq 2 \int_{0}^{\frac{5\varphi}{4} + \frac{3\pi}{8}} \varrho(\cos\chi, \beta V) \, d\chi \leq$$

$$\leq \iint_{\mathcal{V}} (v_1^2 + v_2^2) \, \sigma(|v + \beta V e_2|) \, dv_1 dv_2, \tag{A.37}$$

where $\mathcal{V} = \{v = (v_1, v_2) : |v + \beta V e_2| \geq R\}$ is the complement of the circle of radius R with the center $(0, -\beta V)$. Taking into account that for $v \in \mathcal{V}$, $\beta V \leq |v + \beta V e_2| / \sin(\frac{5\pi}{8} - \frac{5\varphi}{4})$, one has $|v| \leq \beta V + |v + \beta V e_2| \leq |v + \beta V e_2| \, (1 + 1/\sin(\frac{5\pi}{8} - \frac{5\varphi}{4}))$, hence

$$\iint_{\mathcal{V}} (v_1^2 + v_2^2) \, \sigma(|v + \beta V e_2|) \, dv_1 dv_2 \leq$$

$$\leq \left(\frac{1}{\sin(\frac{5\pi}{8} - \frac{5\varphi}{4})} + 1 \right)^2 \iint_{\mathcal{V}} |v + \beta V e_2|^2 \, \sigma(|v + \beta V e_2|) \, dv_1 dv_2 =$$

$$= \left(\frac{1}{\sin(\frac{5\pi}{8} - \frac{5\varphi}{4})} + 1 \right)^2 \cdot 2\pi \int_{R}^{\infty} r^2 \, \sigma(r) \, r dr. \tag{A.38}$$

From (A.37) and (A.38) one obtains

$$\int_{\varphi - \frac{\pi}{2}}^{\frac{3\varphi}{2} + \frac{\pi}{4}} \varrho(\cos\chi, \beta V) \, d\chi \leq \left(\frac{1}{\sin(\frac{5\pi}{8} - \frac{5\varphi}{4})} + 1 \right)^2 \cdot 2\pi \int_{R}^{\infty} r^3 \, \sigma(r) \, dr. \tag{A.39}$$

Further, by hypothesis (b) of lemma 2, for $n > 0$ and for r sufficiently large the function $\gamma(r) := r^{n+3} \sigma(r)$ monotonically decreases, hence for R sufficiently large one has

$$\int_{R}^{\infty} r^{-n} \gamma(r) \, dr \leq \gamma(R) \, \frac{R^{-n+1}}{n-1}, \quad \int_{R/\alpha}^{R} r^{-n} \gamma(r) \, dr \geq \gamma(R)(\alpha^{n-1} - 1) \, \frac{R^{-n+1}}{n-1},$$

thus

$$\int_{R}^{\infty} r^3 \sigma(r) \, dr \leq \frac{1}{\alpha^{n-1} - 1} \int_{R/\alpha}^{R} r^3 \sigma(r) \, dr.$$

By virtue of arbitrariness of n, one concludes that

$$\lim_{R\to+\infty}\frac{\int_R^\infty r^3\sigma(r)\,dr}{\int_{R/\alpha}^R r^3\sigma(r)\,dr}=0. \tag{A.40}$$

From (A.40), (A.36), (A.39), and (A.33) it follows that for any $\varphi\in(\pi/10,\,\pi/2)$

$$\lim_{\beta\to+\infty}\frac{\int_{\varphi-\frac{\pi}{2}}^{\frac{3\varphi}{2}+\frac{\pi}{4}}\varrho(\cos\chi,\beta V)\,d\chi}{\int_{\frac{5\varphi}{4}+\frac{3\pi}{8}}^{\varphi+\frac{\pi}{2}}\varrho(\cos\chi,\beta V)\,d\chi}=0 \tag{A.41}$$

and

$$\int_{\frac{5\varphi}{4}+\frac{3\pi}{8}}^{\varphi+\frac{\pi}{2}}\varrho(\cos\chi,\beta V)\,d\chi\ge c_1(\varphi)\int_{R/\alpha}^R r^3\,\sigma(r)\,dr, \tag{A.42}$$

where $c_1(\varphi)=c\,(1/\sin^2(\frac{5\pi}{8}-\frac{5\varphi}{4})-1)$.

Substituting "$-$" for ε and changing the variable $\chi=\varphi+\theta$ in (A.19), one gets

$$g_-(\varphi,\beta V)=\int_{\varphi-\pi/2}^{\varphi+\pi/2}\varrho(\cos\chi,\beta V)\,\cos(2\chi-3\varphi)\,d\chi. \tag{A.43}$$

Denote $\mathcal{U}_1:=\{v=(v_1,v_2):\,\frac{v_2}{|v_1|}\ge-\tan\varphi\}$, $\mathcal{V}_1:=\{v=(v_1,v_2):\,|v+\beta Ve_2|\ge\beta V\cos\varphi\}$; $\mathcal{V}_1$ is the complement of the circle of radius $\beta V\cos\varphi$ with the center $(0,-\beta V)$. One has $\mathcal{U}_1\subset\mathcal{V}_1$, and for $v\in\mathcal{V}_1$ $|v|\le|v+\beta Ve_2|+\beta V\le|v+\beta Ve_2|\,(1+1/\cos\varphi)$, hence

$$|g_-(\varphi,\beta V)|\le\int_{-\varphi-\pi/2}^{\varphi+\pi/2}\varrho(\cos\chi,\beta V)\,d\chi=\iint_{\mathcal{U}_1}(v_1^2+v_2^2)\,\sigma(|v+\beta Ve_2|)\,dv_1dv_2\le$$

$$\le\left(1+\frac{1}{\cos\varphi}\right)^2\iint_{\mathcal{V}_1}|v+\beta Ve_2|^2\,\sigma(|v+\beta Ve_2|)\,dv_1dv_2=$$

$$=\left(1+\frac{1}{\cos\varphi}\right)^2\cdot2\pi\int_{\beta V\cos\varphi}^\infty r^2\,\sigma(r)\,rdr. \tag{A.44}$$

Taking into account that $\cos(2\chi-3\varphi)\le0$ as $\chi\in[\frac{3\varphi}{2}+\frac{\pi}{4},\,\frac{5\varphi}{4}+\frac{3\pi}{8}]$, $\cos(2\chi-3\varphi)\le\cos\left(\frac{3\pi}{4}-\frac{\varphi}{2}\right)=-\cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)<0$ as $\chi\in[\frac{5\varphi}{4}+\frac{3\pi}{8},\,\varphi+\frac{\pi}{2}]$, and using (A.43), one obtains

$$g_-(\varphi,\beta V)\le\int_{\varphi-\frac{\pi}{2}}^{\frac{3\varphi}{2}+\frac{\pi}{4}}\varrho(\cos\chi,\beta V)\,d\chi-\cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\cdot\int_{\frac{5\varphi}{4}+\frac{3\pi}{8}}^{\varphi+\pi/2}\varrho(\cos\chi,\beta V)\,d\chi. \tag{A.45}$$

From (A.36), (A.41), (A.42), and (A.45) it follows that for any φ and for β sufficiently large

$$g_-(\varphi,\beta V)\le-c_2(\varphi)\int_{R/\alpha}^R r^3\,\sigma(r)\,dr, \tag{A.46}$$

where $R=R(\varphi,\beta)$ and $c_2(\varphi)=\frac{1}{2}\cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\cdot c_1(\varphi)$.

Next, consider $\phi=\phi(\varphi):=\frac{5\varphi}{4}-\frac{\pi}{8}$; one has $\beta V\cos\phi=\beta V\sin\left(\frac{5\varphi}{4}+\frac{3\pi}{8}\right)=R$; hence, by (A.44), for any $\psi\in[0,\,\phi]$

$$|g_-(\psi,\beta V)|\le\left(1+\frac{1}{\cos\psi}\right)^2 2\pi\int_{\beta V\cos\psi}^{\infty}r^3\,\sigma(r)\,dr\le$$

$$\le\left(1+\frac{1}{\cos\phi}\right)^2 2\pi\int_R^{\infty}r^3\,\sigma(r)\,dr. \tag{A.47}$$

Further, one has

$$p'_-(u,\beta V)=\int_0^u p''_-(\nu,\beta V)\,d\nu=2\int_0^{\arctan u}g_-(\psi,\beta V)\,\cos\psi\,d\psi,$$

hence, according to (A.47),

$$\sup_{0\le u\le\tan\phi}|p'_-(u,\beta V)|\le 2\int_0^{\phi}|g_-(\psi,\beta V)|\,d\psi\le$$

$$\le 2\phi\left(1+\frac{1}{\cos\phi}\right)^2 2\pi\int_R^{\infty}r^3\,\sigma(r)\,dr \tag{A.48}$$

and

$$\sup_{u>\tan\phi}\left(-p'_-(u,\beta V)\right)=-\int_0^{\tan\phi}p''_-(\nu,\beta V)\,d\nu+\sup_{u>\tan\phi}\left(-\int_{\tan\phi}^u p''_-(\nu,\beta V)\,d\nu\right)\le$$

$$\le 2\phi\left(1+\frac{1}{\cos\phi}\right)^2 2\pi\int_R^{\infty}r^3\,\sigma(r)\,dr+2(\pi/2-\phi)\cdot\sup_{\psi\in[\phi,\pi/2]}(-g_-(\psi,\beta V)). \tag{A.49}$$

Using (A.31), (A.48), (A.49), and (A.46), one gets that for β sufficiently large

$$0<\frac{P^\beta(\phi)}{\sup_{\psi\in[\phi,\pi/2]}(-g_-(\psi,\beta V))}\le c_3(\varphi)\,\frac{\int_R^\infty r^3\,\sigma(r)\,dr}{\int_{R/\alpha}^R r^3\,\sigma(r)\,dr}+2(\pi/2-\phi), \tag{A.50}$$

where $R=R(\varphi,\beta)$ and $c_3(\varphi):=\frac{4\pi}{c_2(\varphi)}\,\phi\left(1+\frac{1}{\cos\phi}\right)^2$. For arbitrary $\varepsilon>0$, choose φ such that $0<2(\pi/2-\phi(\varphi))<\varepsilon/2$, and then, taking account of (A.40), choose β such that the first term in the right hand side of (A.50) is less that $\varepsilon/2$. The relation (A.32) is established. $\square$

Appendix B

d = 2 Here the formula (2.10) takes the form

$$p_\varepsilon(u,V)=\varepsilon\iint\frac{(v_1u+\varepsilon v_2)_-^2}{1+u^2}\,\rho(v)\,dv_1dv_2,\qquad\varepsilon\in\{-,\,+\}.$$

Passing to the polar coordinates $v = (-r\sin\varphi, -\varepsilon r\cos\varphi)$ and using that $\rho(v) = \sigma(|v + Ve_2|) = \sigma(r) - \varepsilon V\cos\varphi\,\sigma'(r) + o(V),\ V\to 0^+$, one obtains

$$p_\varepsilon(u,V) = \varepsilon\int_0^{2\pi}\int_0^\infty \frac{r^2(\sin\varphi\, u + \cos\varphi)_+^2}{1+u^2}\,(\sigma(r) - \varepsilon V\cos\varphi\,\sigma'(r) + o(V))\,r\,dr\,d\varphi =$$

$$= \varepsilon\int_0^\infty \sigma(r)\,r^3\,dr\cdot I^{(2)} - V\int_0^\infty \sigma'(r)\,r^3\,dr\cdot J^{(2)} + o(V), \tag{B.1}$$

where

$$I^{(2)} = \int_0^{2\pi}(\cos(\varphi-\varphi_0))_+^2\,d\varphi, \qquad J^{(2)} = \int_0^{2\pi}(\cos(\varphi-\varphi_0))_+^2\cos\varphi\,d\varphi,$$

$x_+ := \max\{x, 0\},\ \varphi_0 := \arccos\frac{1}{\sqrt{1+u^2}}$. Changing the variable $\psi = \varphi - \varphi_0$ in these integrals, one obtains

$$I^{(2)} = \int_{-\pi/2}^{\pi/2}\cos^2\psi\,d\psi = \pi/2, \qquad J^{(2)} = \cos\varphi_0\int_{-\pi/2}^{\pi/2}\cos^3\psi\,d\psi = \frac{4}{3\sqrt{1+u^2}}.$$

Substituting the obtained values in (B.1) and using that $-\int_0^\infty \sigma'(r)\,r^3\,dr = 3\int_0^\infty \sigma(r)\,r^2\,dr$, one comes to the formula (4.3.2) with coefficients (4.8).

$\mathbf{d = 3}$ Formula (2.10) takes the form

$$p_\varepsilon(u,V) = \varepsilon\iiint\frac{(v_1u + \varepsilon v_3)_-^2}{1+u^2}\,\rho(v)\,dv_1dv_2dv_3, \qquad \varepsilon\in\{-,\ +\}.$$

Passing to the spherical coordinates $v = (-r\sin\varphi\cos\theta, -r\sin\varphi\sin\theta, -\varepsilon r\cos\varphi)$, one obtains

$$p_\varepsilon(u,V) = \varepsilon\int_0^\pi\int_0^{2\pi}\int_0^\infty\frac{r^2(\sin\varphi\cos\theta\,u + \cos\varphi)_+^2}{1+u^2}\,(\sigma(r) - \varepsilon V\cos\varphi\,\sigma'(r) +$$

$$+o(V))\,r^2\,dr\,d\theta\,\sin\varphi d\varphi = \varepsilon\int_0^\infty\sigma(r)\,r^4\,dr\cdot I^{(3)} - V\int_0^\infty\sigma'(r)\,r^4\,dr\cdot J^{(3)} + o(V), \tag{B.2}$$

where

$$I^{(3)} = \int_0^{2\pi}\int_0^\pi(\cos(\varphi-\phi_0(\theta)))_+^2\,\frac{1+u^2\cos^2\theta}{1+u^2}\,\sin\varphi d\varphi\,d\theta,$$

$$J^{(3)} = \int_0^{2\pi}\int_0^\pi(\cos(\varphi-\phi_0(\theta)))_+^2\,\frac{1+u^2\cos^2\theta}{1+u^2}\,\cos\varphi\sin\varphi d\varphi\,d\theta,$$

$\phi_0(\theta) = \arccos\frac{1}{\sqrt{1+u^2\cos^2\theta}}$. Changing the variable $\psi = \varphi - \phi_0(\theta)$, one obtains

$$I^{(3)} = \int_0^{2\pi}d\theta\,\frac{1+u^2\cos^2\theta}{1+u^2}\,\sin\phi_0(\theta)\int_{-\phi_0(\theta)}^{\pi/2}\cos^3\psi\,d\psi =$$

$$= \int_0^{2\pi} \frac{1+u^2\cos^2\theta}{1+u^2}\ \frac{(1+\sin\phi_0(\theta))^2}{3}\, d\theta =$$

$$= \int_0^{2\pi} \frac{(\sqrt{1+u^2\cos^2\theta}+u\cos\theta)^2}{3(1+u^2)}\, d\theta = \frac{2\pi}{3},$$

$$J^{(3)} = \int_0^{2\pi} d\theta\ \frac{1+u^2\cos^2\theta}{1+u^2} \int_{-\phi_0(\theta)}^{\pi/2} \cos^2\psi\ \frac{\sin(2\psi+2\phi_0(\theta))}{2}\ d\psi =$$

$$= \int_0^{2\pi} d\theta\ \frac{1+u^2\cos^2\theta}{2(1+u^2)} \left[\cos 2\phi_0(\theta) \int_{-\phi_0(\theta)}^{\pi/2} \cos^2\psi\ \sin 2\psi\, d\psi + \right.$$

$$\left. + \sin 2\phi_0(\theta) \int_{-\phi_0(\theta)}^{\pi/2} \cos^2\psi\ \cos 2\psi\, d\psi \right] = \int_0^{2\pi} \frac{1+u\,\phi_0(\theta)\cos\theta}{4(1+u^2)}\ d\theta =$$

$$= \frac{1}{4(1+u^2)} \int_0^{2\pi} [1+(u\cos\theta)\ \arctan(u\cos\theta)]\, d\theta = \frac{\pi}{2\sqrt{1+u^2}}.$$

Substituting these values in (B.2) and using that $-\int_0^\infty \sigma'(r)\, r^4\, dr = 4\int_0^\infty \sigma(r)\, r^3\, dr$, one gets the formula (4.3.2) with coefficients (4.9).

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